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Slow Relaxations and Bifurcations of the Limit Sets of Dynamical Systems. II. Slow Relaxations of a Family of Semiflows

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Abstract—We propose a number of approaches to the notion of the relaxation time of a dynamical system which are motivated by the problems of chemical kinetics, give exact mathematical definitions of slow relaxations, study their possible reasons, among which an important role is played by bifurcations of limit sets.

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Key words: *compact metric spaces, dynamical systems, slow relaxations, bifurcations of limit sets*

To the Memory of Sergei L'vovich Sobolev

In the first part of our work [1], we have studied bifurcations of the limit sets of dynamical systems. Here we suggest six approaches to the notion of relaxation time, discuss the interrelationship between them, study their relations with bifurcations of limit sets and other peculiarities of the dynamics. Of course, we preserve all definitions and the notation of [1] and continue the numeration of sections, assertions, formulas, and figures.

5. RELAXATION TIMES

The main object of study in this article is the relaxation time of a dynamical system. We call so the time in which the motion of the system in a sense attains the limit regime. Fix a point $x \in X$, a parameter $k \in K$, and a "boundary precision" $\varepsilon > 0$.

Put

$$\tau_1(x,k,\varepsilon) = \inf\{t > 0 \mid \rho^*(f(t,x,k),\omega(x,k)) < \varepsilon\},$$
$$\tau_2(x,k,\varepsilon) = \mathrm{mes}\{t > 0 \mid \rho^*(f(t,x,k),\omega(x,k)) \geq \varepsilon\},$$
$$\tau_3(x,k,\varepsilon) = \inf\{t > 0 \mid \rho^*(f(t',x,k),\omega(x,k)) < \varepsilon \ \text{ for } \ t' > t\},$$
$$\eta_1(x,k,\varepsilon) = \inf\{t > 0 \mid \rho^*(f(t,x,k),\omega(k)) < \varepsilon\},$$
$$\eta_2(x,k,\varepsilon) = \mathrm{mes}\{t > 0 \mid \rho^*(f(t,x,k),\omega(k)) \geq \varepsilon\},$$
$$\eta_3(x,k,\varepsilon) = \inf\{t > 0 \mid \rho^*(f(t',x,k),\omega(k)) < \varepsilon \ \text{ for } \ t' > t\}.$$

Here ρ^* stands for the distance from a point to a set, namely: if $x \in X$ and $P \subset X$ then $\rho^*(x,P) = \inf\{\rho(x,y) \mid y \in P\}$. As for relaxation times, figuratively speaking, τ_1 (η_1) is the moment of the first appearance of the (x,k)-motion in the ε-neighborhood of the set of (all) its ω-limit points, τ_2 (η_2) is the time of stay of the (x,k)-motion outside the ε-neighborhood of the corresponding limit set, and τ_3 (η_3) is the moment of the ultimate entry in this neighborhood.

Proposition 11. *For each of the three values of i, the quantities τ_i and η_i are well defined, finite, and $\tau_i \geq \eta_i$. Here $\tau_1 \leq \tau_2 \leq \tau_3$ and $\eta_1 \leq \eta_2 \leq \eta_3$.*

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Proof. As above, fix $x \in X$, $k \in K$, and $\varepsilon > 0$. Whatever the six quantities we are interested in are, their definitions are correct for $i \neq 2$. For $i = 2$, it is about the measures of "Lebesgue sets" of the functions $\rho^*(f(t, x, k), \omega(x, k))$ and $\rho^*(f(t, x, k), \omega(k))$ continuous in t. These sets are measurable; and, hence, our definitions of τ_2 and η_2 are also correct.

As for the above inequalities, we note only that they are obvious. Indeed, the first three follow from the relation $\omega(x, k) \subset \omega(k)$. The remaining two pairs of inequalities may be commented as follows: the moment of the first appearance in the ε-neighborhood of the set of limit points is preceded by the period of stay outside this neighborhood but it lasts no longer than the moment of the ultimate entry therein.

We see now that τ_3 is always the greatest of the six quantities. We are left with proving that τ_3 is finite. For this, we only need to verify that $\rho^*(f(t, x, k), \omega(x, k)) < \varepsilon$ for all sufficiently large $t > 0$. Suppose on the contrary that there exists an infinitely large sequence of t_n for which $\rho^*(f(t_n, x, k), \omega(x, k)) \geq \varepsilon$ for all n. The compactness of X enables us to assume that the sequence $f(t_n, x, k)$ has a limit. By definition, this limit is an ω-limit point of the (x, k)-motion though it lies outside the set $\omega(x, k)$ at the distance at least ε therefrom. Thus, τ_3 is defined as the infimum of a nonempty set and, therefore, is finite. The proposition is proved. □

Definition 4. We say that a family of semiflows has τ_i-*slow relaxation* if $\tau_i(x, k, \varepsilon)$ is not bounded above on $X \times K$ for some $\varepsilon > 0$. Families of semiflows with η_i-slow relaxation are defined similarly.

In the remaining part of the section, we will speak of individual semiflows, naturally, excluding from the notation any reference to the index, which could take only one value. Of course, also in this case, our definitions of different types of slow relaxations make sense.

Proposition 12. *No individual semiflow can have* η_1*-slow relaxation.*

Proof. Suppose on the contrary that, for a semiflow $f(t, x)$, there are $\varepsilon > 0$ and a sequence x_n such that $\eta_1(x_n, \varepsilon) \to \infty$. Since X is compact, we may assume that x_n converges to some $x^* \in X$. Show that $\rho^*(f(t, x^*), \omega(k)) > \varepsilon/2$ for all $t > 0$ and so $\eta_1(x^*, \varepsilon/2)$ is infinite, which contradicts Proposition 11.

Given $\eta > 0$, the mapping f is uniformly continuous on the product $[0, \eta] \times X$. Thus, there is $\delta > 0$ such that $\rho(f(t, x), f(t, x^*)) < \varepsilon/2$ whenever $0 \leq t \leq \eta$ and $\rho(x, x^*) < \delta$. Since $\eta_1(x_n, \varepsilon) \to \infty$ and $x_n \to x^*$, there exists a number n such that $\eta_1(x_n, \varepsilon) > \eta$ and $\rho(x_n, x^*) < \delta$. The former inequality means that $\rho^*(f(t, x_n), \omega(k)) \geq \varepsilon$ for $0 \leq t \leq \eta$, and the latter enables us to assert that, for the same t, the distance between $f(t, x_n)$ and $f(t, x^*)$ is less than $\varepsilon/2$. This obviously implies the desired $\rho^*(f(t, x^*), \omega(k)) > \varepsilon/2$, valid for all $t > 0$ since the number $\eta > 0$ was chosen arbitrarily. The proposition is proved. □

It is easy to understand that an analogous assertion holds also for a family of semiflows provided that it possesses some property like equicontinuity guaranteed, for example, by the compactness of the set of indices. Of course, this fails for the most general families. On the other hand, the other five types of slow relaxation may well occur and do occur in individual flows, of which we will become convinced with the help of simple concrete examples. Figures 3 and 4 demonstrate the possibilities that arise here.

Example 3 (η_2-slow relaxations). In the circle $x^2 + y^2 \leq 1$ on the cartesian plane (x, y), consider the system defined by the following equation in polar coordinates

$$\dot{r} = r(r-1)(r\cos\varphi + 1), \qquad \dot{\varphi} = r\cos\varphi + 1. \tag{8}$$

Here the full ω-limit set consists of the two fixed points: $r = 0$ and $r = 1$, $\varphi = \pi$ (Fig. 3, a).

Example 4 (η_3-slow relaxations not belonging to the type η_2). We slightly modify the previous example by replacing the boundary loop by an unstable limit cycle:

$$\dot{r} = r(r-1), \qquad \dot{\varphi} = 1. \tag{9}$$

Now the full ω-limit set includes the entire boundary circle and the point $r = 0$ (Fig. 3, b). The time of stay outside its ε-neighborhood is bounded for every ε. However, $\eta_3(r, \varphi, 1/2) \to \infty$ if $r \to 1$.

Example 5 (τ_1-slow relaxations not belonging to the types η_2 and η_3). In the annulus $1/2 \leq x^2 + y^2 \leq 1$, consider the system defined by differential equations in polar coordinates:

$$\begin{aligned} \dot{r} &= (1-r)(r\cos\varphi + 1)(1 - r\cos\varphi), \\ \dot{\varphi} &= (r\cos\varphi + 1)(1 - r\cos\varphi). \end{aligned} \tag{10}$$

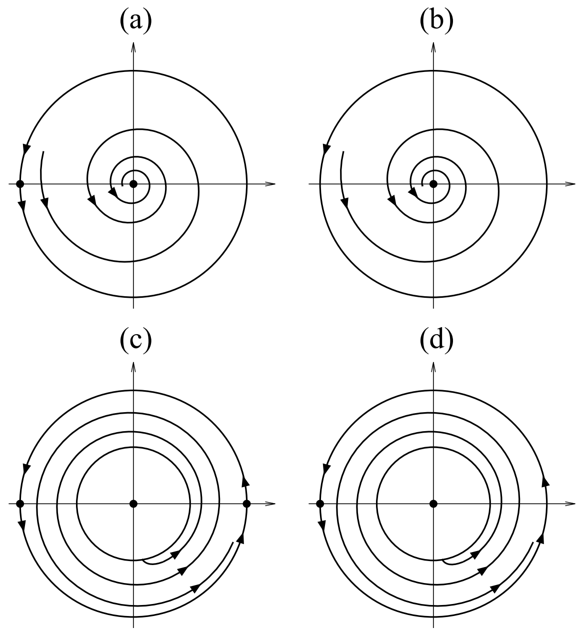

Fig. 3. The phase portraits of the systems: (a) — (8); (b) — (9); (c) — (10); and (d) — (11)

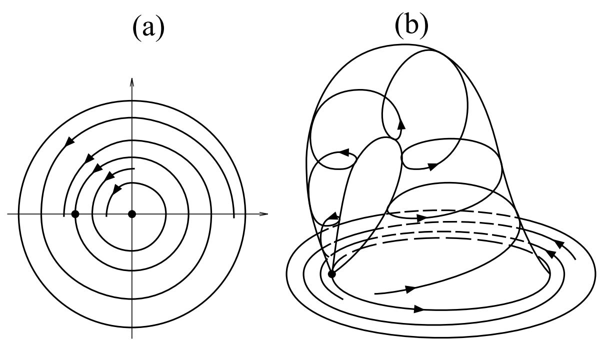

Fig. 4. The phase portrait of (12): (a) without the identification (gluing) of the fixed points; (b) after the identification

In this case, the full ω-limit set is the entire circle $r = 1$ (Fig. 3, c). Note that $\tau_1(r, \varphi, 1/2) \to \infty$ as $r = 1$, $\varphi \to \pi$, and $\varphi > \pi$, since, for such (r, φ), the set $\omega(r, \varphi)$ consists of the point $r = 1$, $\varphi = 0$.

Example 6 (τ_3-slow relaxations not belonging to the types τ_1, τ_2, and η_3). Simplify the previous example by leaving only one equilibrium point on the boundary circle $r = 1$, namely, let

$$\begin{aligned} \dot{r} &= (1 - r)(r \cos\varphi + 1), \\ \dot{\varphi} &= r \cos\varphi + 1. \end{aligned} \tag{11}$$

Now, $\tau_3(r, \varphi, 1/2) \to \infty$ as $r = 1$, $\varphi \to \pi$ but $\varphi > \pi$. Moreover, the times τ_1 and τ_2 remain bounded for every fixed $\varepsilon > 0$, since, as is shown in Fig. 3, d, for the just-indicated points (r, φ), the set $\omega(r, \varphi)$ consists of the point $r = 1$, $\varphi = \pi$. It remains to note that the functions η_2 and η_3 are bounded. This is due to the fact that the full ω-limit set of our system is the circle $r = 1$.

Example 7 (τ_2-slow relaxations not belonging to the types τ_1 and η_2). We did not find an elementary example on the plane without using Lemma 2 [1]. Consider first the semiflow in the disk $x^2 + y^2 \leq 2$ defined by the equations

$$\begin{aligned} \dot{r} &= -r(1 - r)^2[(r \cos\varphi + 1)^2 + r^2 \sin^2\varphi], \\ \dot{\varphi} &= (r \cos\varphi + 1)^2 + r^2 \sin^2\varphi. \end{aligned} \tag{12}$$

There are three types of limit sets $\omega(r, \varphi)$ in this system: the circle $r = 1$ if $r > 1$; the point $r = 1$, $\varphi = \pi$ if $r = 1$; and the origin $r = 0$ if $r < 1$.

This is seen very well in Fig. 4, a. Now, make the "topological identification" of the points $r = 0$ and $r = 1$, $\varphi = \pi$ as is shown in the "extremely symbolic" Fig. 4, b. Clearly, in the new system, $\tau_2(r, \varphi, 1/2) \to \infty$ as $r \to 1$, $r < 1$, although τ_1 remains bounded as well as η_2. Note however that η_3 is unbounded.

Most of the examples above are noncoarse dynamical systems, which is not accidental. As we will see in the third part of our article, in coarse dynamical systems on the usual Euclidean plane, slow relaxations τ_1, τ_2, τ_3, and η_3 can occur only simultaneously.

6. SLOW RELAXATIONS AND BIFURCATIONS OF THE LIMIT SETS

In the simplest cases, the relationship between slow relaxations and bifurcations of limits sets is obvious. It is worth mentioning the situation when, on its way to the limit set, the motion is "delayed" near an unstable equilibrium of the system under study. These delays lead, on the one hand, to an arbitrarily slow process of relaxation of motion and, on the other, for quite obvious reasons, to the violation of the continuous dependence of the limit set on the initial conditions or to its bifurcation. The study of the question is complicated also by the fact that we have several approaches to the notion of slow relaxation. Moreover, as will be proved below, the bifurcations are not the only source of the slow relaxations. However, the relationship between bifurcations and slow relaxations is obvious for the time of the first appearance of the trajectory in a given neighborhood of the limit set.

In what follows, we assume not only the compactness of the space X but also that of the set of parameters K although some of our assertions hold without this extra assumption.

Theorem 1. *A system has τ_1-slow relaxation if and only if it admits $\Omega(x, k)$-bifurcations.*

Proof. (1) Let (x^*, k^*) be one of the $\Omega(x, k)$-bifurcation points of the system under consideration. This means that there exists a sequence (x_n, k_n) in $X \times K$ converging to (x^*, k^*) for which $\Omega(x_n, k_n)$ do not D-converge to $\Omega(x^*, k^*)$. In other words, there exist $\varepsilon > 0$ and $x' \in X$ such that $\omega(x', k^*) \subset \omega(x^*, k^*)$ and $r(\omega(x', k^*), \omega(x_n, k_n)) > \varepsilon$ for arbitrarily large n. Passing to an appropriate subsequence, we may assume that the last inequality holds for all numbers.

Suppose that $y^* \in \omega(x', k^*)$. Then $y^* \in \omega(x^*, k^*)$, and, so, we can find a sequence $t_m > 0$ tending to infinity for which $f(t_m, x^*, k^*) \to y^*$. Since $f(t_m, x_n, k_n)$ tends to $f(t_m, x^*, k^*)$ as $n \to \infty$ for each m; therefore, there exists an infinitely large sequence $n(m)$ such that $f(t_m, x_{n(m)}, k_{n(m)}) \to y^*$ as $m \to \infty$. In order not to introduce new denotations and to simplify those already arisen, assign the number m to the points $x_{n(m)}$ and $k_{n(m)}$ and then simply replace m by n. This enables us to assume in the sequel that the points $y_n = f(t_n, x_n, k_n)$ tend to y^* as $n \to \infty$. Moreover, obviously, $\omega(y_n, k_n) = \omega(x_n, k_n)$ and, so, $r(\omega(x', k_n), \omega(y_n, k_n)) > \varepsilon$ for all n.

Now, fix arbitrary $\tau > 0$. The fact that $(y_n, k_n) \to (y^*, k^*)$ as $n \to \infty$ and the uniform continuity of f on compact sets yield $\rho(f(t, y^*, k^*), f(t, y_n, k_n)) < \varepsilon/2$ for all t in the interval $0 \le t \le \tau$ if n is large enough. Assuming so t and n and involving the fact that, after y^*, the point $f(t, y^*, k^*)$ belongs to $\omega(x', k^*)$, we come to

$$\rho^*(f(t, y^*, k^*), \omega(y_n, k_n)) > \varepsilon,$$

whence

$$\rho^*(f(t, y_n, k_n), \omega(y_n, k_n)) > \varepsilon/2.$$

Thus, $\tau_1(y_n, k_n, \varepsilon/2) \ge \tau$, and this means the τ_1-slow relaxation of our system. The first part of the theorem is proved.

(2) Conversely, suppose that the system has τ_1-relaxation. In other words, for some $\varepsilon > 0$, there exists a sequence (x_n, k_n) in $X \times K$ such that $\tau_1(x_n, k_n, \varepsilon) \to \infty$ as $n \to \infty$. The compactness of X and K enables us to assume it convergent. Show that its limit (x^*, k^*) is an $\Omega(x, k)$-bifurcation point for the system.

In $\omega(x^*, k^*)$, choose a point y and prove that $\rho^*(y, \omega(x_n, k_n)) > \varepsilon/2$ for all sufficiently large n. First of all, wait until a moment $t^* > 0$ with $\rho(f(t^*, x^*, k^*), y) < \varepsilon/4$. Since $(x_n, k_n) \to (x^*, k^*)$ and $\tau_1(x_n, k_n, \varepsilon) \to \infty$, there exists a number $n(y)$ such that

$$\rho(f(t^*, x^*, k^*), f(t^*, x_n, k_n)) < \varepsilon/4, \quad \tau_1(x_n, k_n, \varepsilon) > t^*$$

for all $n \geq n(y)$. Note that, in addition, $\rho^*(f(t^*, x_n, k_n), \omega(x_n, k_n)) > \varepsilon$ by the definition of τ_1, and come to the desired conclusion:

$$\rho^*(f(t^*, x^*, k^*), \omega(x_n, k_n)) > 3\varepsilon/4$$

and, hence, $\rho^*(y, \omega(x_n, k_n)) > \varepsilon/2$ for all $n \geq n(y)$.

Now, find an $\varepsilon/4$-net $y_1, \dots, y_m$ for the compact set $\omega(x^*, k^*)$ and take as N the greatest of the numbers $n(y_1), \dots, n(y_m)$. The inequality $\rho^*(y_l, \omega(x^*, k^*)) > \varepsilon/2$ holds for all $l = 1, \dots, m$ and $n \geq N$. As is easy to understand, in this case, $\rho^*(y, \omega(x^*, k^*)) > \varepsilon/4$ already for every $y \in \omega(x^*, k^*)$ and the same $n \geq N$. In other words, $r(\omega(x^*, k^*), \omega(x_n, k_n)) \geq \varepsilon/4$ when $n \geq N$, and this, by Proposition 7 [1], means the existence of $\Omega(x, k)$-bifurcations for our system. The proof of the theorem is complete. □

From the just-proven Theorem 1 and Proposition 9 [1] it follows that the system admits a τ_1-slow relaxation if and only if $\omega(x, k)$ as a mapping from $X \times K$ into $B(X)$ does not satisfy the r-continuity condition.

The following theorem describing the connection between η_1-relaxations and $\Omega(k)$-bifurcations is analogous to the previous one, which we will stress in every way by the scheme of its proof.

Theorem 2. *A system has η_1-slow relaxation if and only if it admits $\Omega(k)$-bifurcations.*

Proof. (1) Let k^* be one of the $\Omega(k)$-bifurcation points of the system. This means that there is a sequence k_n in K converging to k^* for which $\Omega(k_n)$ do not D-converge to $\Omega(k^*)$. In other words, there are $\varepsilon > 0$ and $x^* \in X$ such that $r(\omega(x^*, k^*), \omega(k_n)) > \varepsilon$ for arbitrarily large n. Of course, we may assume last inequality for all n. Now, take a point $x_0 \in \omega(x^*, k^*)$ and prove that $\eta_1(x_0, k_n, \varepsilon/2) \to \infty$ as $n \to \infty$.

We observe first that $\rho^*(f(t, x_0, k^*), \omega(k_n)) > \varepsilon$ for all n and $t > 0$ since $f(t, x_0, k^*) \in \omega(x^*, k^*)$. Given $\eta > 0$, find $\delta(\eta) > 0$ such that

$$\rho(f(t, x_0, k^*), f(t, x_0, k_n)) < \varepsilon/2 \quad \text{if} \quad 0 \leq t \leq \eta, \qquad \rho_K(k^*, k_n) < \delta(\eta).$$

Since $k_n \to k^*$, there exists $N(\eta)$ such that $\rho_K(k_n, k^*) < \delta(\eta)$ for all $n \geq N(\eta)$. Therefore,

$$\rho^*(f(t, x_0, k_n), \omega(k_n)) > \varepsilon/2, \qquad t \in [0, \eta].$$

Hence, $\eta_1(x_0, k_n, \varepsilon/2) \geq \eta$ for $n \geq N(\eta)$, so that the system has η_1-slow relaxation.

(2) Conversely, suppose that the system has some η_1-slow relaxation. In other words, for some $\varepsilon > 0$, there exists a sequence (x_n, k_n) in $X \times K$ such that $\eta_1(x_n, k_n, \varepsilon) \to \infty$ as $n \to \infty$. The compactness of X and K makes it possible to assume the sequence convergent. Show that its limit (x^*, k^*) is an $\Omega(k)$-bifurcation point of the system.

Choose a point y in $\omega(x^*, k^*)$ and prove that $\rho^*(y, \omega(k_n)) > \varepsilon/2$ for all sufficiently large n. We first wait by the moment $t^* > 0$, when $\rho(f(t^*, x^*, k^*), y) < \varepsilon/4$. Since

$$(x_n, k_n) \to (x^*, k^*), \qquad \eta_1(x_n, k_n, \varepsilon) \to \infty,$$

there is a number $n(y)$ such that

$$\rho(f(t^*, x^*, k^*), f(t^*, x_n, k_n)) < \varepsilon/4, \qquad \eta_1(x_n, k_n, \varepsilon) > t^*$$

for all $n \geq n(y)$. Hence we obtain what was required: $\rho^*(y, \omega(k_n))$ is estimated from below by

$$\rho^*(f(t^*, x_n, k_n), \omega(k_n)) - \rho(f(t^*, x^*, k^*), y) - \rho(f(t^*, x^*, k^*), f(t^*, x_n, k_n)),$$

where the first summand is at least ε and the moduli of the second and the third summands are less than $\varepsilon/4$, so that $\rho^*(y, \omega(k_n)) > \varepsilon/2$ for all $n \geq n(y)$.

Further, the arguments related to the construction of an $\varepsilon/4$-net (but now for the set $\omega(k^*)$) that we know from the previous theorem, lead to the inequality $r(\omega(x^*, k^*), \omega(k_n)) > \varepsilon/4$ for sufficiently large n. By Proposition 7 [1], this means that k^* is an $\Omega(k)$-bifurcation point of the system. The theorem is proved. □

Pass to the study of the relationship between slow relaxations and the bifurcations "of the second kind." As we will see, here bifurcations guarantee slow relaxation as well but the system may pass to the limit regime for a very long time and without ruptures of limit sets.

Lemma 3. *For every points $x_0 \in X$ and $k \in K$ and numbers $\varepsilon > \delta > 0$, there exists $t_0 > 0$ such that the conditions $\rho(x, x_0) < \delta$ and $0 \le t' \le t_0$ imply $\rho(x_0, f(t', x, k)) < \varepsilon$.*

Proof. Suppose the contrary; namely, let there exist sequences x_n and t'_n such that $\rho(x_0, x_n) < \delta$ and $t'_n \to 0$ but $\rho\big(x_0, f\big(t'_n, x_n, k\big)\big) \ge \varepsilon$. By the compactness of X, we may assume that x_n converges to some $x_* \in X$. The function $\rho(x_0, f(t, x, k))$ is jointly continuous. Therefore,

$$\rho\big(x_0, f\big(t'_n, x_n, k\big)\big) \to \rho(x_0, f(0, x_*, k)) = \rho(x_0, x_*).$$

Since $\rho(x_0, x_n) < \delta$, we have $\rho(x_0, x_*) \le \delta$. This contradicts the inequalities

$$\rho\big(x_0, f\big(t'_n, x_n, k\big)\big) \ge \varepsilon > \delta.$$

The lemma is proved. □

Theorem 3. *A system admitting $\omega(x, k)$-bifurcations necessarily has τ_2-slow relaxation.*

Proof. Let (x^*, k^*) be an $\omega(x, k)$-bifurcation point. Then there exist a number $\varepsilon > 0$, a sequence (x_n, k_n) converging to (x^*, k^*), and some $x' \in \omega(x^*, k^*)$ such that $\rho^*(x', \omega(x_n, k_n)) > \varepsilon$ for all n.

Since $x' \in \omega(x^*, k^*)$, there exists an infinitely large sequence of $t_j > 0$ such that $\rho(f(t_j, x^*, k^*), x') < \varepsilon/8$ for all j. As is easy to understand, Lemma 3 implies the existence of $t_0 > 0$ such that the inequality $\rho(f(\tau + t_j, x^*, k^*), x') < \varepsilon/4$ holds for each j and all τ in $0 \le \tau \le t_0$. Here, of course, we should recall that f stands for a semiflow and, hence, $f(\tau + t_j, x^*, k^*) = f(\tau, f(t_j, x^*, k^*))$. Now that t_0 is chosen, "rarefy" the sequence t_j so that, after its natural renumeration, we have $t_{j+1} - t_j > t_0$.

To each $t > 0$, assign the auxiliary quantity

$$\theta(x^*, x', t, \varepsilon) = \mathrm{mes}\,\{t' \mid 0 \le t' \le t,\ \rho(f(t', x^*, k^*), x') < \varepsilon/4\}$$

showing us how long the (x^*, k^*)-trajectory stays in the $(\varepsilon/4)$-neighborhood of x' in the time period from zero to t. In the situation under discussion, obviously, $\theta(x^*, x', t, \varepsilon) > jt_0$ if $t > t_j + t_0$. For each j, find a number $N(j)$ so that the inequality $\rho(f(t, x_n, k_n), f(t, x^*, k^*)) < \varepsilon/4$ holds when $n \ge N(j)$ and $0 \le t \le t_j + t_0$. If $n \ge N(j)$ then $\rho(f(t, x_n, k_n), x') < \varepsilon/2$ as t ranges over any of the intervals $t_i \le t \le t_i + t_0$, where $1 \le i \le j$. Hence, $\tau_2(x_n, k_n, \varepsilon/2) > jt_0$ for $n \le N(j)$, which in turn means that our system has τ_2-slow relaxation. The proof of the theorem is complete. □

Theorem 4. *A system admitting $\omega(k)$-bifurcations necessarily has η_2-slow relaxation.*

Proof. Let k^* be an $\omega(k)$-bifurcation point. Then there exist a number $\varepsilon > 0$, a sequence of "parameters" k_n converging to some $k^* \in K$, and a point $x' \in \omega(k^*)$ such that $\rho^*(x', \omega(x_n, k_n)) > \varepsilon$ for all n. To agree with that, the reader only needs to recall the corresponding definitions. Suppose that a point $x^* \in X$ is such that $x' \in \omega(x^*, k^*)$. Now, choose an arbitrary $\tau > 0$ and choose $t^* > 0$ such that $\theta(x^*, x', t^*, \varepsilon) > \tau$. The function θ was defined in the proof of the previous theorem. The existence of the above moment was justified there as well. Involving the uniform continuity of f on compact sets, we may assert that $\rho(f(t, x^*, k^*), f(t, x^*, k_n)) < \varepsilon/4$ for sufficiently large n and all t with $0 \le t \le t^*$. This implies the estimate

$$\eta_2(x^*, k_n, \varepsilon/2) > \theta(x^*, x', t, \varepsilon) > \tau.$$

The arbitrariness of $\tau > 0$ witnesses the η_2-slow relaxation of the system. The theorem is proved. □

The following two theorems give us other conditions guaranteeing slow relaxations of the types τ_2 and η_2. Their meaning is that the presence of a quite "complicated" motion in the system already guarantees its slow relaxation.

Theorem 5. *If the α-limit set of some entire motion is not included in the ω-limit set then the system has τ_2-slow relaxation.*

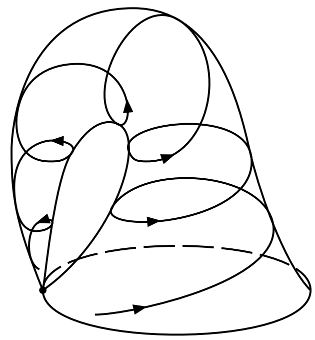

Fig. 5. The phase portrait of (8) after the identification (gluing) of fixed points

Proof. Let x^* be an α-limit point of an entire (x, k)-motion but does not belong to its ω-limit set. By the compactness of $\omega(x, k)$, the distance $\rho^*(x^*, \omega(x, k))$ between it and x^* is greater than zero. Choose $\varepsilon > 0$ strictly less than this distance and, for every $t > 0$, define the auxiliary quantity

$$\varphi(x, x^*, t, \varepsilon) = \text{mes}\,\{t' \mid -t \leq t' \leq 0,\ \rho(f(t', x, k), x^*) < \varepsilon/2\}.$$

Since this quantity is less than $\tau_2(f(-t, x, k), k, \varepsilon/2)$, it suffices to prove that it tends to infinity as $t \to \infty$.

By Lemma 3, there exists $t_0 > 0$ such that $\rho(f(t, y, k), x^*) < \varepsilon/2$ if $0 \leq t \leq t_0$ and $\rho(y, x^*) < \varepsilon/4$. Since $x^* \in \alpha(x, k)$, there exists a sequence t_n tending to $-\infty$ such that $\rho(f(t_n, x, k), x^*) < \varepsilon/4$. We may assume that here $t_{n+1} - t_n < -t_0$. In this case, like in the proof of Theorem 3, we obtain the estimate $\varphi(x, x^*, t_n, \varepsilon) > nt_0$. The theorem is proved. □

Theorem 6. *If the limit set $\alpha(x, k)$ of some entire (x, k)-motion of the system is not included in the closure of $\omega(k)$ then the system has η_2-slow relaxation.*

Proof. Suppose that the (x, k)-motion is entire. Choose $x^* \in \alpha(x, k)$ not belonging to the closure of $\omega(k)$. Then the distance ε from x^* to this closure is strictly positive. Define $\varphi(x, x^*, t, \varepsilon)$ as in the proof of the previous theorem. Here it tends to ∞ as $t \to \infty$ too. We are left with observing that

$$\eta_2(f(-t, x, k), k, \varepsilon/2) > \varphi(x, x^*, t, \varepsilon).$$

The theorem is proved. □

The aim of the following slightly "abstract" example is to show that the hypotheses of the last two theorems and, hence, their claims may be realized without bifurcations of the limit sets.

Example 8. Consider the already familiar system of Example 3 whose phase portrait is depicted in Fig. 4 but now, as we did it in Example 7, identify formally its fixed points $r = 0$ and $r = 1$, $\varphi = \pi$. The full ω-limit set of the so-obtained system (as is seen in Fig. 5 similar to Fig. 4, b) consists of a single fixed point; and so bifurcations are absolutely out of question here. However, for the initial data (r_0, φ_0), where $r_0 < 1$ and $r_0 \to 1$, and φ_0 is arbitrary, the relaxation time $\eta_2(r_0, \varphi_0, 1/2)$ and $\tau_2(r_0, \varphi_0, 1/2)$ therewith tend to infinity.

Before passing to the analysis of slow relaxations of types τ_3 and η_3, we, following [2, p. 363], define the notion of stability of motion in the sense of Poisson. We note right away that each motion of this kind is automatically entire.

Definition 5. Say that the (x, k)-motion is positively stable in the sense of Poisson or, in short, P^+-stable if $x \in \omega(x, k)$.

Lemma 4. *A system admitting an entire but not P^+-stable motion has τ_3-slow relaxation.*

Proof. If the entire (x, k)-motion is not P^+-stable then the distance from x to the compact set $\omega(x, k)$ is strictly greater than zero. We are left with observing that $\tau_3(f(-t, x, k), k, \varepsilon) \geq t$ for all $t > 0$, since $f(t, f(-t, x, k), k) = x$; and, moreover, $\rho^*(x, \omega(f(-t, x, k), k) = \varepsilon$, since $\omega(f(-t, x, k), k) = \omega(x, k)$. Thus, the system has τ_3-slow relaxation. The proof is over. □

Lemma 5. *If there exists an entire (x, k)-motion such that x does not belong to the closure of $\omega(k)$ then the system has η_3-slow relaxation.*

Proof. Take as ε the distance from x to the closure of $\omega(k)$ and observe that $\eta_3(f(-t, x, k), k, \varepsilon) \geq t$ for all $t > 0$ for the same reasons as those indicated in the proof of the previous lemma. Thus, the system has η_3-slow relaxation. The proof is over. □

Lemma 6. *Consider the entire (x_0, k_0)-motion of a system that has no $\omega(x, k)$-bifurcation. Then, for every $x^* \in \alpha(x_0, k_0)$, the set $\omega(x^*, k_0)$ is included in the intersection $\alpha(x_0, k_0) \cap \omega(x_0, k_0)$.*

Proof. Let $x^* \in \alpha(x_0, k_0)$. Find a sequence of $t_n > 0$ such that

$$t_n \to \infty, \qquad x_n = f(-t_n, x_0, k_0) \to x^* \quad \text{as} \quad n \to \infty.$$

Here it is important to stress that $\omega(x_n, k_0) = \omega(x_0, k_0)$ for all n. If $\omega(x^*, k_0)$ were a part of the closed set $\omega(x_0, k_0)$ then we would obtain the inequality

$$d(\omega(x^*, k_0), \omega(x_n, k_0)) \equiv d(\omega(x^*, k_0), \omega(x_0, k_0)) > 0$$

meaning that the sequence $\omega(x_n, k_0)$ does not tend to $\omega(x_0, k_0)$ in the sense of d-convergence although $x_n \to x^*$. In other words, (x^*, k_0) would be an $\omega(x, k)$-bifurcation point. Thus, $\omega(x^*, k_0) \subset \omega(x_0, k_0)$. We are left with observing that the closure of the (x^*, k_0)-trajectory, on the one hand, includes $\omega(x^*, k_0)$, and on the other, entirely lies in the closed set $\alpha(x_0, k_0)$. The lemma is proved. □

Theorem 7. *A system has τ_3-slow relaxation if and only if it satisfies at least one of the following conditions:*

(1) *there exist $\omega(x, k)$-bifurcations;*

(2) *there exists a P^+-stable motion.*

Proof. The sufficiency of either of the two conditions for the τ_3-slow relaxation of the system follows from Theorem 3 and Lemma 4 respectively. Suppose conversely that the system has some τ_3 -slow relaxation. Assume that it has no $\omega(x, k)$-bifurcations. We only need to construct its entire motion that is not positively stable in the sense of Poisson. We first find $\varepsilon > 0$ and (x_n, k_n) such that

$$\tau_n = \tau_3(x_n, k_n, \varepsilon) \to \infty.$$

The compactness of X and K enables us to assume this sequence convergent to some (x^*, k^*). Put $y_n = f(\tau_n, x_n, k_n)$ and note that $\rho^*(y_n, \omega(x_n, k_n)) = \varepsilon$, as it follows from the definition of the relaxation time τ_3 and the continuous dependence of the distance on the variable point to a fixed set. Of course, we may assume that the sequence y_n is also convergent. Let y^* be its limit. Prove that the (y^*, k^*)-motion has the desired properties. First, it is entire. This is obvious from Proposition 2 [1] if we take into account the fact that the (y_n, k_n)-motion is defined on the time interval from $-\tau_n$ to $+\infty$. Second, it is not Poisson stable "to the right." Indeed, assume that $y^* \in \omega(y^*, k^*)$. Since $y_n \to y^*$, it follows that $\rho(y^*, y_n) < \varepsilon/2$ for all sufficiently large n. Then, for the same n, we infer that $\rho^*(y^*, \omega(y_n, k_n)) > \varepsilon/2$, since $\rho^*(y_n, \omega(y_n, k_n)) = \varepsilon$. But this and our assumption that y^* is included in $\omega(y^*, k^*)$ imply

$$d(\omega(y^*, k^*), \omega(y_n, k_n)) > \varepsilon/2;$$

and, hence, at (y^*, k^*), there happens an $\omega(x, k)$-bifurcation of the system. The theorem is proved. □

By Lemma 6, the second condition of Theorem 7 may be replaced as follows: *there exists an entire and not P^+-stable (x, k)-motion of the system such that $\omega(x^*, k) \subset \omega(x, k)$ for all $x^* \in \alpha(x, k)$.* Some examples are given by trajectories going from a fixed point to the same point like a loop of a separatrix or the homoclinic trajectories of periodic motions.

Theorem 8. *A system has η_3-slow relaxation if and only if it satisfies at least one of the conditions:*

(1) *there exist $\omega(k)$-bifurcations;*

(2) *there exists an entire (y^*, k^*)-motion for which y^* does not lie in the closure of $\omega(k^*)$.*

Proof. The logical scheme of arguments here is the same as in the proof of the preceding theorem. We first note that the sufficiency of each of the two conditions for η_3-slow relaxation of the system follows from Theorem 4 and Lemma 5 respectively. Suppose conversely that the system has η_3-slow relaxation. Assume that it has no $\omega(k)$-bifurcations. We are left with constructing a motion of the system meeting condition (2) of [1]. To this end, find a number $\varepsilon > 0$ and a sequence (x_n, k_n) such that

$$\theta_n = \eta_3(x_n, k_n, \varepsilon) \to \infty.$$

We may assume that it converges to some point (x^*, k^*). Put $y_n = f(\theta_n, x_n, k_n)$ and observe that $\rho^*(y_n, \omega(k_n)) = \varepsilon$. We may assume also that y_n converges to some y^*. Prove that the (y^*, k^*)-motion has the desired properties. First, it is entire: we already know the arguments. Second, y^* does not belong to the closure $\overline{\omega}(k^*)$ of the set $\omega(k^*)$. Indeed, $y_n \to y^*$, and so $\rho^*(y^*, \overline{\omega}(k_n)) > \varepsilon/2$ for all sufficiently large n, since, obviously, $\rho^*(y_n, \overline{\omega}(k_n)) = \varepsilon$. Thus, if $y^* \in \overline{\omega}(k^*)$ then the inequality $d(\omega(k^*), \omega(k_n)) > \varepsilon/2$ holds for large n; hence, at k^*, there happens an $\omega(k)$-bifurcation of the system. The proof of Theorem 8 is complete. □

Corollary 2. *Suppose that, for every $k \in K$, any trajectory in $\omega(k)$ is recurrent and there are no entire non-P^+-stable (x, k)-motions such that $\omega(x^*, k) \subset \omega(x, k)$ for all $x^* \in \alpha(x, k)$ or, weaker, $\alpha(x_0, k_0) \cap \omega(x_0, k_0) \neq \varnothing$. Then the notions of τ_i-slow relaxation of the system are equivalent for* $i = 1, 2, 3$.

This is obvious from Theorem 7 and Proposition 11. We stress that, by one of the Birkhoff theorems [3] (see also [2, p. 402]), the condition of the recurrence of each trajectory in $\omega(k)$ is fulfilled in the case when $\omega(x, k)$ is minimal for every x.

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