Workshop: Principal manifolds for data cartography and dimension reduction

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Invariant manifolds in the dimension reduction for fluid conveying tubes

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Motivation

In the seventies and eighties of the last century considerable progress has been achieved in the treatment of stability and bifurcation of solutions of non-linear low-dimensional dynamical and statical systems. This progress was connected with names like *V.I.Arnol'd, J.Moser, R.Thom, S.Smale, M.Golubitsky*, and many others.

For eample Takens-Bogdanov bifurcation at a nilpotent double zero eigenvalue results in the following Normal Form

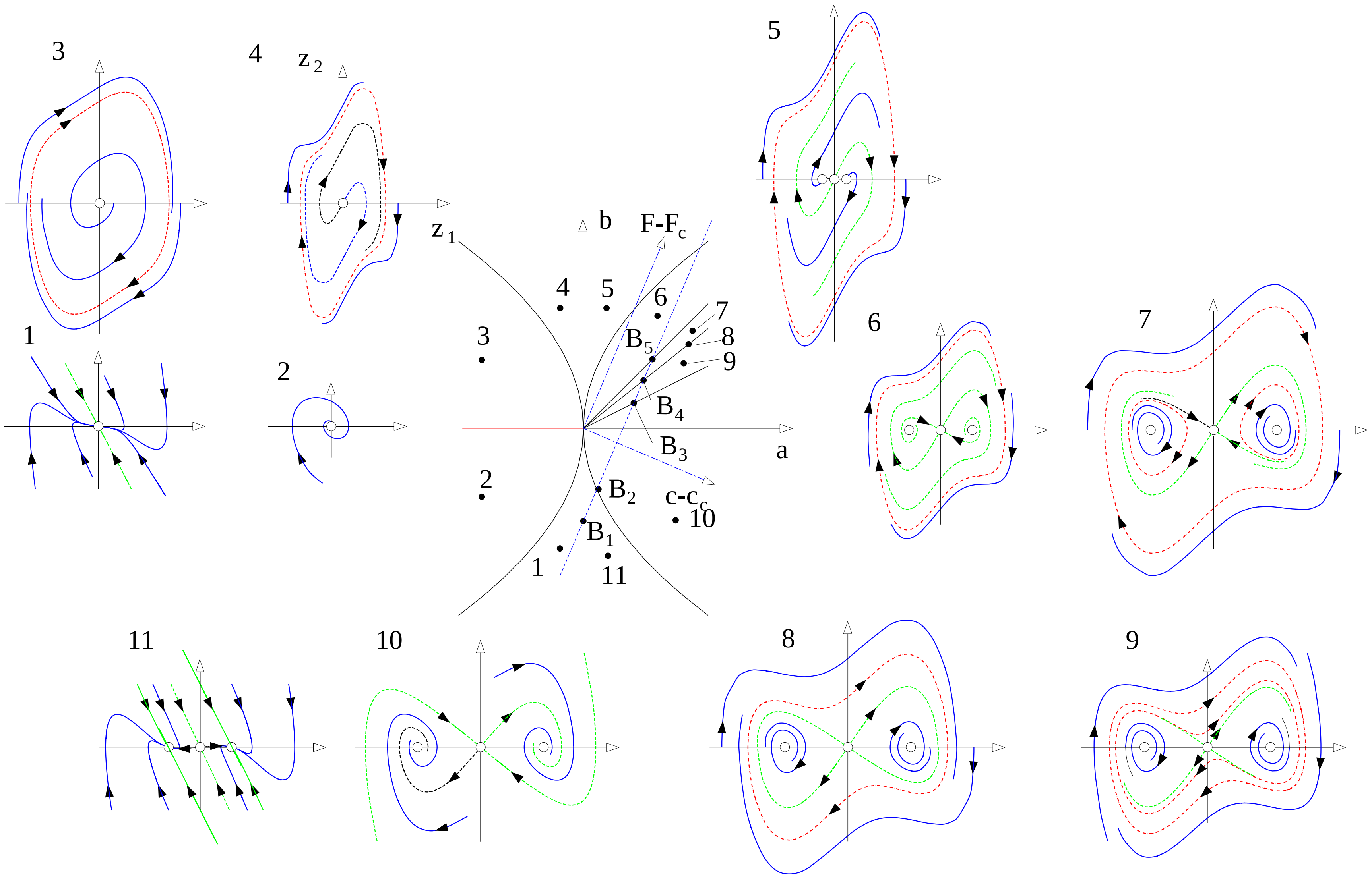
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= Ax_1^3 + Bx_1^2 x_2 \end{aligned}$$

which is equivalent to the second order equation

$$\ddot{x} - Ax^3 - Bx^2\dot{x} - ax - b\dot{x} = 0$$

where two unfolding terms with the parameter values a, b have been introduced.

The following diagramm is obtained for $A < 0$ and $B < 0$.



One might wonder what this progress in Applied Mathematics might have to do with practical problems encountered, for example, in stability investigations in Mechanics or Engineering. In Engineering both statical and dynamical problems are typically modelled by systems with a large or even infinite number of degrees of freedom whereas, for example, in the books by *Arnol'd, Guckenheimer and Holmes, Golubitsky and Schaeffer or Iooss and Joseph* only very low dimensional systems are analysed.

The connection between low dimensional dynamical systems and infinte or large finite dimensional systems, if there is any at all, may be given by the concept of dimension-reduction.

Is there a theory of dimension reduction?

There is no general theory of dimension reduction.

But it is well known that under certain conditions dimension reduction is possible. Then the next questions arise:

1. Under which conditions is dimension reduction possible?

2. To what extend does the reduced system still keep the essential features and properties of the full system?

What is dimension reduction?

There are various different forms of dimension reduction in Mechanics.

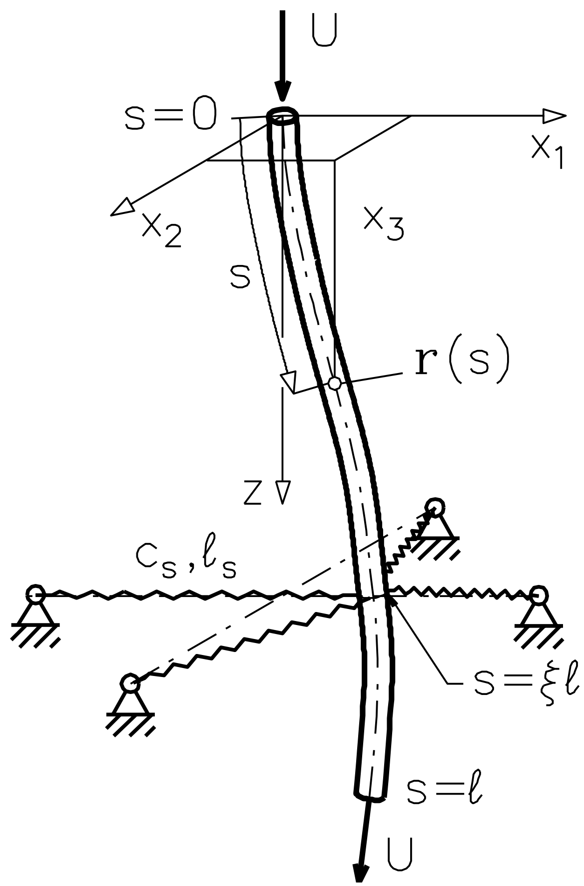
All of them is common that in one way or the other the quantities, introduced or obtained for the description of the system, may be strongly reduced in their number, because it turns out that some of them are not essential for the (asymptotic) behavior of the considered problem.

This fact is well known either:

- form engineering experience (experiments) or
- from numerical simulation or
- from rigorous mathematical proof.

We consider the fundamental problem of stability theory: The loss of stability of an equilibrium position of a high dimensional or infinite dimensional system under quasistatic variation of a system parameter.

Typical example: Loss of stability of the downhanging position of a fluid conveying tube.



Mathematically the problem can be posed as follows:

Consider an infinite dimensional system given by a partial differential equation with boundary conditions

$$\boldsymbol{G}\left(\frac{\partial}{\partial t}, \nabla, \boldsymbol{v}(\boldsymbol{x}, t), \lambda\right) = \boldsymbol{0} \ ,$$

including a parameter λ.

We want to derive the finite dimensional system

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{f}(\boldsymbol{q}, \lambda) \qquad \boldsymbol{q} \in \mathbb{R}^m \ .$$

Its solution should allow to approximate the field $\boldsymbol{v}(\boldsymbol{x}, t)$ in some appropriate norm.

Two basic questions: (1) What are the $q_i(t)$?
(2)How to find the finite dimensional system?

We rewrite the field equation in the form

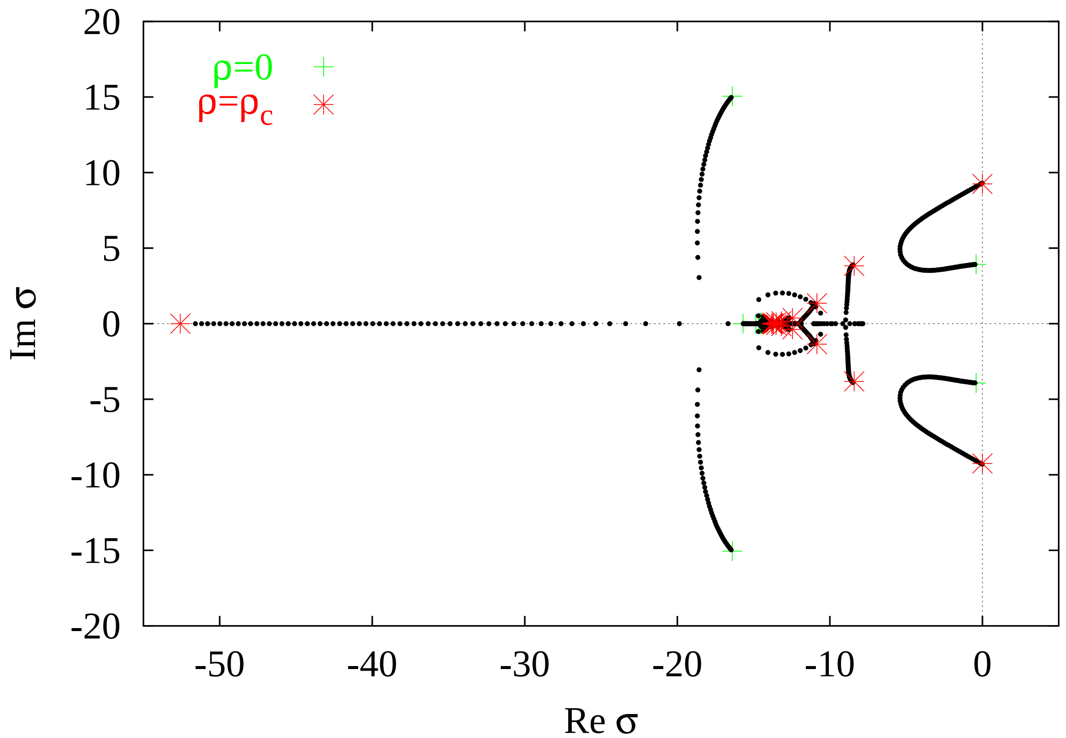
$$\dot{\boldsymbol{u}} = \boldsymbol{A}(\lambda)\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{u}, \lambda),$$

- $\boldsymbol{A} = \boldsymbol{G}_v(\boldsymbol{v}_e)$ is the linearization of the operator $\boldsymbol{G}$ at the equilibrium position $\boldsymbol{v}_e$,
- $\boldsymbol{g}$ is a smooth nonlinear operator
- $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_e$ is the deviation from $\boldsymbol{v}_e$.

Both operators $\boldsymbol{A}$ and $\boldsymbol{g}$ still depend on the spatial variable $\boldsymbol{x}$. In addition, we assume that $\boldsymbol{g}(\mathbf{0}, \lambda) = \mathbf{0}$ and $\boldsymbol{g}_u(\mathbf{0}, \lambda) = \mathbf{0}$.

The first step in treating the stability problem of the tube is looking at the linearized problem and calculating the eigenvalues and the corresponding eigenfunctions (eigenvectors).

For quasistatically increasing the flow rate ρ we depict the movement of the eigenvalues σ in the complex plane.



It can be seen that at $\rho = \rho_c$ a pair of complex conjugate eigenvalues crosses the imaginary axis whereas all other eigenvalues have a negative realpart.

The loss of stability can be described in terms of the temporal evolution of the amplitudes of certain (active) modes that are mildly unstable or only slightly damped in linear theory (*Coullet and Spiegel 1983*).

If the number of the active modes is finite, their amplitudes are governed by a set of ordinary differential equations: *the amplitude equations of the critical modes*.

The $q_i(t)$ are the amplitudes of the active modes and the m ordinary differential equations are the amplitude equations describing their time evolution.

These amplitude equations govern, at least locally, the behavior of the full system since the other (infinitely many) modes are damped and don't appear in the description.

Hence the following task must be solved:

1. Search for preferred patterns (coherent structures)
2. Find a simple model which describes the dynamics or the interaction of the coherent structures

How to find the active modes and to calculate the amplitude equations?

There are no general answers how to identify the active modes.

However, if we restrict to the szenario of loss of stability of an equilibrium for a quasistatic variation of a parameter λ, some (mathematically sound) answers can be given (Aceves, A., Adachihara, H., Jones, C., Lerman, J. C., McLaughlin, D. W., Moloney, J. V., Newell, A. C., 1986).

We distinguish three cases:

(I) $|\lambda - \lambda_c| = O(\varepsilon)$: values of λ are close to the transition value λ_c .

(II) $|\lambda - \lambda_c| = O(1)$: moderate deviations of λ from λ_c.

(III) $|\lambda - \lambda_c| = O(1/\varepsilon)$, large deviations of λ from λ_c.

How to find a reduced order system?

1. Local theory:
   (a) Dynamics: Center manifold theory
   (b) Statics: Liapunov-Schmidt method

2. Global theory:
   (a) Theoretically important: Inertial manifold theory
   (b) Practically important: Galerkin methods (Approximate inertial manifold theory)

3. Slow-fast dynamics in Hamiltonian systems (Nonlinear Normal Modes).

4. Linear Static Problems: Condensation

Local theory

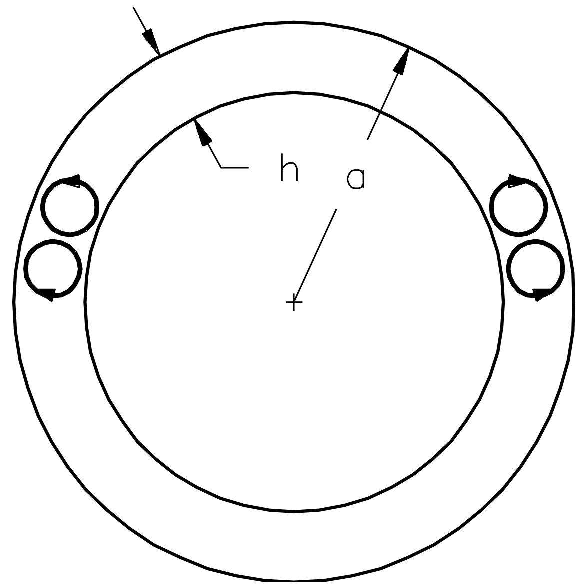
We must still further distinguish between small and large aspect ratio systems.

The aspect ratio decides whether a strong straight forward dimension reduction to amplitude equations can be performed.

The aspect ratio refers to the ratio of the length scale of the extension of the system to the length scale which is characteristic for the dynamics of the problem.

Example: Benard-problem in spherical geometry (*Chossat 1979*)

If the fluid layer has thickness h and the radius of the sphere is a then the multiplicity of the critical eigenvalue at loss of stability is:



- $\frac{h}{a} = 1.0$ 3-fold
- $\frac{h}{a} = 0.7$ 5-fold
- $\frac{h}{a} \to 0$ ∞-fold.

Hence the planar Benard problem has an eigenvalue with infinite multiplicity at loss of stability. This is easy to understand, because for two wave numbers k_x and k_y only one equation is given

$$k_x^2 + k_y^2 = k^2,$$

where k is determined from the neutral stability curve.

Analogous problem: Buckling of a complete spherical shell under uniform compression

Here due to the high multiplicity of the critical eigenvalue a reduction of the infinite dimensional problem to a finite dimensional problem is only possible if many modes are retained.

Numerical results are presented in Hoff, Madsen, Mayers (1966) and Yamaki (1984) for elastic buckling of an axially compressed circular cylindrical shell. Here > 50 terms must be included to obtain a sufficiently accurate result.

Local theory: Center manifold theory

Literature: *Carr 1981, Holmes 1981, Coullet & Spiegel, 1983.*

The field variable $\boldsymbol{u}(\boldsymbol{x},t)$ given in the Hilbert space $\boldsymbol{E}$ is decomposed in the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_c(\boldsymbol{x},t) + \boldsymbol{u}_s(\boldsymbol{x},t) = \sum_{i=1}^{m} q_i(t)\boldsymbol{w}_i(\boldsymbol{x}) + \sum_{j=m+1}^{\infty} q_j(t)\boldsymbol{w}_j(\boldsymbol{x}), \tag{1}$$

where the $\boldsymbol{w}_i(\boldsymbol{x})$ are the active modes and $\boldsymbol{w}_j(\boldsymbol{x})$ the passive (stable) modes, with j typically ranging to infinity. The modes are obtained from the solution of the eigenvalue problem related to the linear system

$$\dot{\boldsymbol{u}} = \boldsymbol{A}(\lambda_c)\boldsymbol{u}. \tag{2}$$

We assume that the spectrum of $\boldsymbol{A}(\lambda)$ is discrete and that for $\lambda = \lambda_c$ a finite number (m) of eigenvalues crosses the imaginary axis at the same time. All other eigenvalues have a negative real part.

We rewrite the field equation in the form

$$\begin{aligned}\dot{\boldsymbol{u}}_c &= P\boldsymbol{A}\boldsymbol{u}_c + P\boldsymbol{g}(\boldsymbol{u}_c + \boldsymbol{u}_s),\\ \dot{\boldsymbol{u}}_s &= Q\boldsymbol{A}\boldsymbol{u}_s + Q\boldsymbol{g}(\boldsymbol{u}_c + \boldsymbol{u}_s),\end{aligned} \tag{3}$$

by decomposing the Hilbert space $\boldsymbol{E} = \boldsymbol{E}_c \oplus \boldsymbol{E}_s$, where $\boldsymbol{E}_c$ is finite (m)-dimensional and $\boldsymbol{E}_s$ is closed.

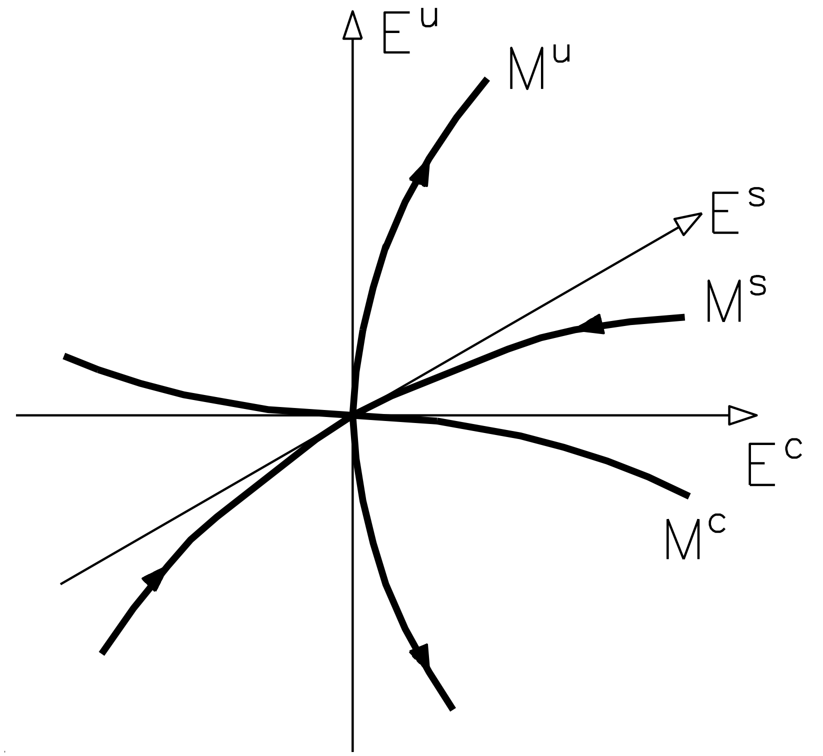
P is the projection onto $\boldsymbol{E}_c$ along $\boldsymbol{E}_s$, giving $\boldsymbol{u}_c = P\boldsymbol{u} \in \boldsymbol{E}_c$ and $\boldsymbol{u}_s = Q\boldsymbol{u} \in \boldsymbol{E}_s$ where $Q = I - P$.

If $\boldsymbol{u}_s = \boldsymbol{h}(\boldsymbol{u}_c)$ is a smooth invariant manifold we call $\boldsymbol{h}$ a *center manifold* if $\boldsymbol{h}(\mathbf{0}) = \boldsymbol{h}'(\mathbf{0}) = \mathbf{0}$.

Note that if in (3) $P\boldsymbol{g} = Q\boldsymbol{g} = \mathbf{0}$, all solutions tend exponentially fast to solutions of $\dot{\boldsymbol{u}}_c = P\boldsymbol{A}\boldsymbol{u}_c$. The linear m-dimensional equation on the (flat) center manifold determines the asymptotic behavior of the entire linear infinite-dimensional system.

The *center manifold theorem* enables us to extend this argument to the nonlinear case when $P\boldsymbol{g}$ and $Q\boldsymbol{g}$ are not equal to zero.

Linear eigen-spaces E^u, E^c, E^s and invariant manifolds M^u, M^c, M^s of the nonlinear system



Center Manifold Theorem 1. *1. There exists a center manifold $\boldsymbol{u}_s = \boldsymbol{h}(\boldsymbol{u}_c)$ for system (3) if $|\boldsymbol{u}_c|$ is sufficiently small. The behavior of (3) on the center manifold is governed by the equation*

$$\dot{\boldsymbol{u}}_c = P\boldsymbol{A}\boldsymbol{u}_c + P\boldsymbol{g}(\boldsymbol{u}_c + \boldsymbol{h}(\boldsymbol{u}_c)). \tag{4}$$

2. The zero solution of (3) has exactly the same stability properties as the zero solution of (4).

3. The center manifold can be calculated from

$$\mathbf{h}'(\boldsymbol{u}_c)(P\boldsymbol{A}\boldsymbol{u}_c + P\mathbf{g}(\boldsymbol{u}_c, \mathbf{h}(\boldsymbol{u}_c)) = Q\boldsymbol{A}\mathbf{h}(\boldsymbol{u}_c) + Q\mathbf{g}(\boldsymbol{u}_c, \mathbf{h}(\boldsymbol{u}_c)) \ .$$

4. An approximation $\boldsymbol{H} : R^m \to R^{n_s}$ of $\mathbf{h}$ is a smooth map with $\boldsymbol{H}(\mathbf{0}) = \boldsymbol{H}'(\mathbf{0}) = \mathbf{0}$

and is defined by the equation

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{H}) := \boldsymbol{H}'(\boldsymbol{u}_c)[P\boldsymbol{A}\boldsymbol{u}_c + P\boldsymbol{g}(\boldsymbol{u}_c + \boldsymbol{H}(\boldsymbol{u}_c))] \\ - Q\boldsymbol{A}\boldsymbol{H}(\boldsymbol{u}_c) - Q\boldsymbol{g}(\boldsymbol{u}_c + \boldsymbol{H}(\boldsymbol{u}_c)),\end{aligned} \tag{5}$$

If

$$\boldsymbol{R}(\boldsymbol{H}) = \mathcal{O}(|\boldsymbol{u}_c|^r), \quad r > 1, \quad \textit{as} \quad |\boldsymbol{u}_c| \to 0,$$

we have $|\boldsymbol{h}(\boldsymbol{u}_c) - \boldsymbol{H}(\boldsymbol{u}_c)| = \mathcal{O}(|\boldsymbol{u}_c|^r)$ *as* $|\boldsymbol{u}_c| \to 0$.

Inserting $\boldsymbol{u}_s = \boldsymbol{h}(\boldsymbol{u}_c)$ into $(3)_1$ eliminates the (infinitely many) inessential variables $\boldsymbol{u}_s$ to obtain (4), a system of m nonlinear ordinary differential equations for the m amplitudes $q_i(t)$ of the active modes $\boldsymbol{w}_i(\boldsymbol{x})$.

(4) describes the whole nearby (local) dynamics of the original infinite dimensional system (2).

The $\boldsymbol{u}_s$ in (1) are at least of second order in $\boldsymbol{u}_c$, that is

$$\boldsymbol{u}_s = \mathcal{O}(\|\boldsymbol{u}_c\|^2) = \mathcal{O}(|q_i|^2). \tag{6}$$

Relation (6) has important consequences concerning the practical calculations.

Part (3) of the theorem allows to calculate a sufficiently accurate approximation by retaining relevant terms in a Taylor series expansion.

Example (*Carr 1981*): Stability of the equilibrium $\boldsymbol{y}_0 = \mathbf{0}$ of the system

$$\begin{aligned} \dot{y}_1 &= y_1 y_2 + a y_1^3 + b y_1 y_2^2 \\ \dot{y}_2 &= -y_2 + c y_1^2 + d y_1^2 y_2. \end{aligned}$$

The linear part is in diagonalized form. The eigenvalues are 0 and -1. Thus, y_1 is the critical and y_2 the non-critical variable.

Setting $y_2 = 0$ the first equation would take the form

$$\dot{y}_1 = a y_1^3 \ .$$

However, this is incorrect. We must calculate the center manifold of the form

$$y_2 = H(y_1) = \alpha_2 y_1^2 + \alpha_3 y_1^3 + \alpha_4 y_1^4 + \dots \quad .$$

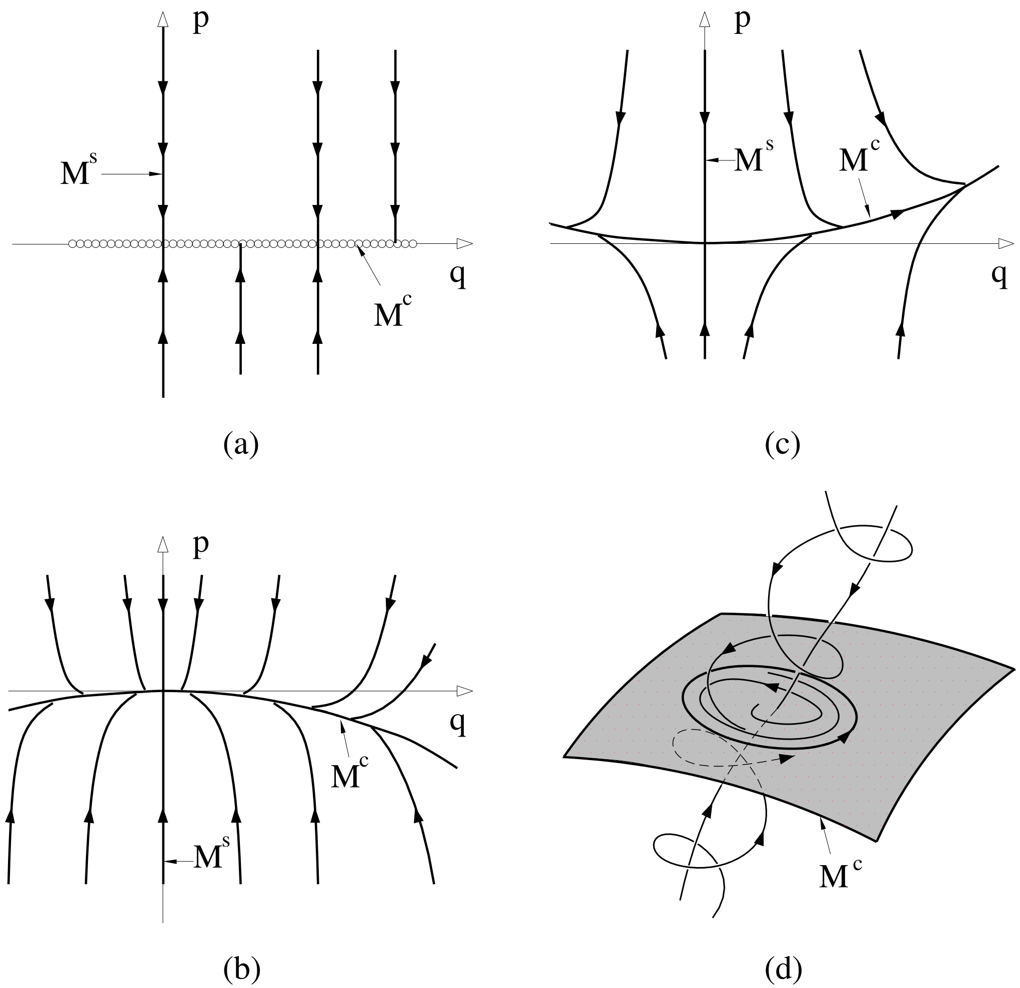
from Point (3) of the Theorem.
Up to second order terms the CM follows to

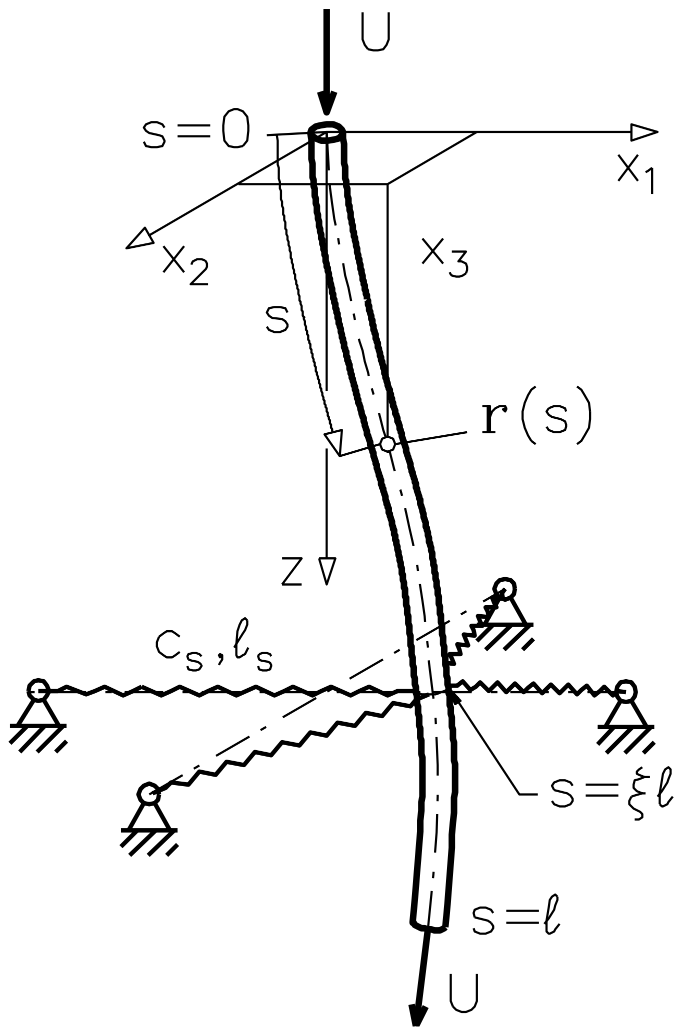
$$y_2 = h(y_1) = c y_1^2 + O(|y_1|^3).$$

yielding

$$\dot{y}_1 = (a + c) y_1^3 \ .$$

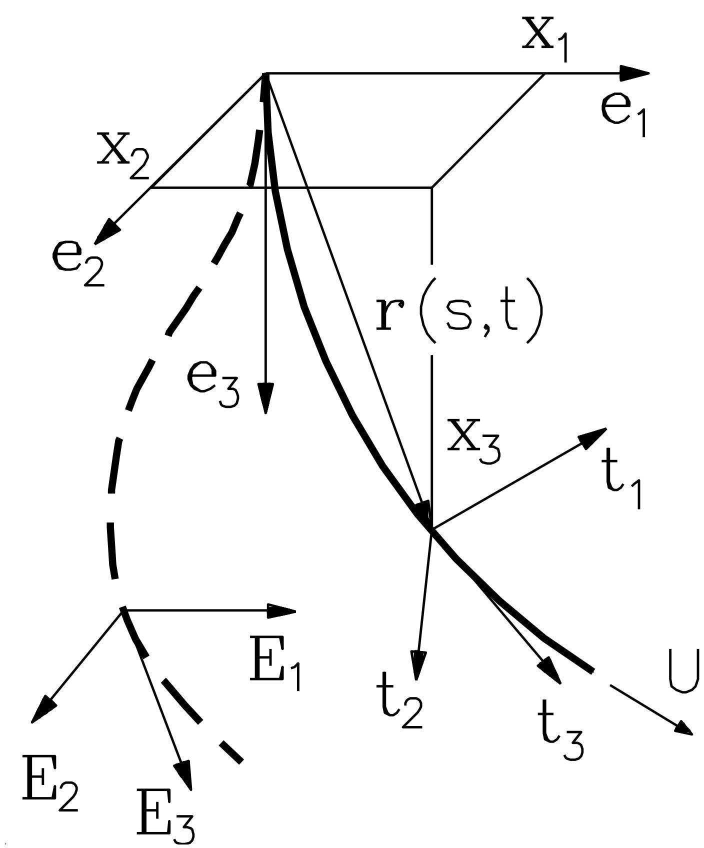
$a = b = d = 0$: (a) $c = 0$, (b) $c < 0$, (c)$c > 0$





U
s=0
x1
x2
x3
s
r(s)
z
c_s, ℓ_s
s=ξℓ
s=ℓ
U

Geometry of the tube



Equations of motion of the tube

The radius vector $\boldsymbol{r}(s,t)$ and second the rotation matrix $\mathbf{B}(s,t) \in \mathbf{SO}(3)$ define the orientation of the cross-section of the tube given by the vectors $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\boldsymbol{t}_3$ with respect to a space fixed triad $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

With the notations $(\ )' = \partial/\partial s$ and $(\ )^{\cdot} = \partial/\partial t$ we obtain the first kinematic relationship $\boldsymbol{t}_3 = \boldsymbol{r}' = \mathbf{B}\boldsymbol{e}_3$.

The second is a skew symmetric infinitesimal rotation matrix $\widehat{\boldsymbol{\Omega}}$ which follows from $\mathbf{B}$ to

$$\widehat{\boldsymbol{\Omega}}(s,t) = \mathbf{B}^{-1}\mathbf{B}' = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} ,$$

Ω_1 and Ω_2 measure bending of the rod about the axes $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ and Ω_3 measures the twist about $\boldsymbol{t}_3$.

By $\widehat{\boldsymbol{\Omega}}$ a vector $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$ is defined. From the linear and angular momentum principles we obtain two more vector equations. The full set of nonlinear partial differential equations of motion in dimensionless variables $(s \to s/\ell)$ reads

$$
\begin{aligned}
\boldsymbol{r}' &= \mathbf{B}\boldsymbol{e}_3 \\
\mathbf{B}' &= \mathbf{B}\widehat{\boldsymbol{\Omega}} \\
\boldsymbol{T}' &= \boldsymbol{T} \times \boldsymbol{\Omega} + \boldsymbol{F} \times \boldsymbol{e}_3 - \mathbf{B}^T \boldsymbol{m}_0 \\
\boldsymbol{F}' &= \boldsymbol{F} \times \boldsymbol{\Omega} + \mathbf{B}^T(-\gamma \boldsymbol{e}_3 + \alpha_e \dot{\boldsymbol{r}} + \ddot{\boldsymbol{r}} + 2\sqrt{\beta}\varrho \dot{\boldsymbol{r}}' + \varrho^2 \boldsymbol{r}'' - \boldsymbol{q}_0).
\end{aligned}
$$

$\boldsymbol{F}$ is the resultant force in the cross-section and $\boldsymbol{m}_0$ and $\boldsymbol{q}_0$ are distributed loadings. Due to the inextensibility constraint and the neglection of shear deformation there exists no constitutive relationship involving $\boldsymbol{F}$ and only one between the moment vector $\boldsymbol{T} = (T_1, T_2, T_3)^T$ and $\boldsymbol{\Omega}$ in the form

$$
\begin{aligned}
T_1 &= \Omega_1 + \alpha_1 \dot{\Omega}_1 \ , \\
T_2 &= \Omega_2 + \alpha_1 \dot{\Omega}_2 \ , \\
T_3 &= \gamma_3(\Omega_3 + \alpha_3 \dot{\Omega}_3),
\end{aligned}
$$

where $\gamma_3 = GJ_T/EJ$ is a quotient between the torsional and bending stiffnesses and the α_i are material damping coefficients.

Our numerical investigations are based on numerical data taken from (Sugiyama,Tanaka,Kishi,Kawagoe 1985).

ϱ is the dimesionless flow rate proportional to the velocity U,

Boundary conditions at $s = 0$ and $s = 1$

$$\boldsymbol{r}(0) = \mathbf{0}\ , \qquad \mathbf{B}(0) = \mathbf{E}\ , \qquad \boldsymbol{F}(1) = \mathbf{0}\ , \qquad \boldsymbol{T}(1) = \mathbf{0}\ ,$$

and the jump condition for $\boldsymbol{F}$ at the location ξ of the elastic support

$$\boldsymbol{F}(\xi_+) - \boldsymbol{F}(\xi_-) = -\mathbf{B}^T(\xi)\boldsymbol{f}_s,$$

where $\boldsymbol{f}_s$ is the restoring force of the spring.

Eulerian angles to represent the rotation matrix results in singular situations at the trivial downhanging configuration.

Following Bajaj, Sethna (1984) and Buzano, Geymonat, Poston (1985) we use the coordinates x_1, x_2 and an angle describing the twist as variables.
In these variables for four springs $\boldsymbol{f}_s$ is given up to third order terms by

$$\boldsymbol{f}_s = -c \begin{pmatrix} x_1 + \alpha_s \dot{x}_1 + \varepsilon_s^2 \left(\frac{x_1}{2}(x_1^2 - 4x_2^2) + \alpha_s \left[(x_1^2 - x_2^2)\dot{x}_1 - 2x_1 x_2 \dot{x}_2 \right] \right) \\ x_2 + \alpha_s \dot{x}_2 + \varepsilon_s^2 \left(\frac{x_2}{2}(x_2^2 - 4x_1^2) - \alpha_s \left[(x_1^2 - x_2^2)\dot{x}_2 + 2x_1 x_2 \dot{x}_1 \right] \right) \\ 0 \end{pmatrix}.$$

$c = 4\ell^3 c_s/(2EJ)$ is the dimensionless spring stiffness,

α_s a damping constant and $\varepsilon_s = \ell/\ell_s$ where ℓ_s is the length of a spring,

$\boldsymbol{f}_s$ becomes rotationally symmetric for $\ell_s \to \infty$.

Hence a rotationally symmetric support can be approximately realized either by an array of springs or very long springs.

Steps in the computation of the amplitude equations

1. Solution of linear eigenvalue problem and of the adjoint problem since the linear operator $\mathbf{A}$ is not self-adjoint.

2. Definition of projection on Null space and on the space orthogonal to it.

3. Calculation of an approximation of the center manifold up to the necessary order following from Determinacy of the amplitude equations.

4. In the tube problem, if third order terms were sufficient, point (3) is trivial. In this case we have

$$\hat{\boldsymbol{u}}(s,t) = \sum_{j=1}^{m} q_j(t)\boldsymbol{w}_j(s)$$

The system of m amplitude equations is ($\boldsymbol{w}_k^\star$ is the adjoint eigenfunction to $\boldsymbol{w}_k$)

$$\int_0^1 \boldsymbol{w}_k^{\star T}\dot{\hat{\boldsymbol{u}}}ds = \int_0^1 \boldsymbol{w}_k^{\star T}\mathbf{A}\hat{\boldsymbol{u}}ds + \int_0^1 \boldsymbol{w}_k^{\star T}\mathbf{g}(\hat{\boldsymbol{u}},\lambda_c)ds \ , \qquad k = 1,\ldots,m.$$

Linearization about trivial state

Linearization of equations of motion about the trivial straight downhanging state

$$\boldsymbol{r} = s\boldsymbol{e}_3 \ , \qquad \mathbf{B} = \mathbf{E} \ , \qquad \boldsymbol{T} \equiv \mathbf{0} \ , \qquad \boldsymbol{F} = (1-s)\gamma\boldsymbol{e}_3$$

which is a solution for all values of ϱ and c results in

$$\ddot{x}_i + \delta\dot{x}_i + x_i^{IV} + \alpha_1\dot{x}_i^{IV} + 2\sqrt{\beta}\varrho\dot{x}_i' + \varrho^2 x_i'' - \gamma[(1-s)x_i']' = 0$$

for $i = 1, 2$ and one equation for the twisting angle χ

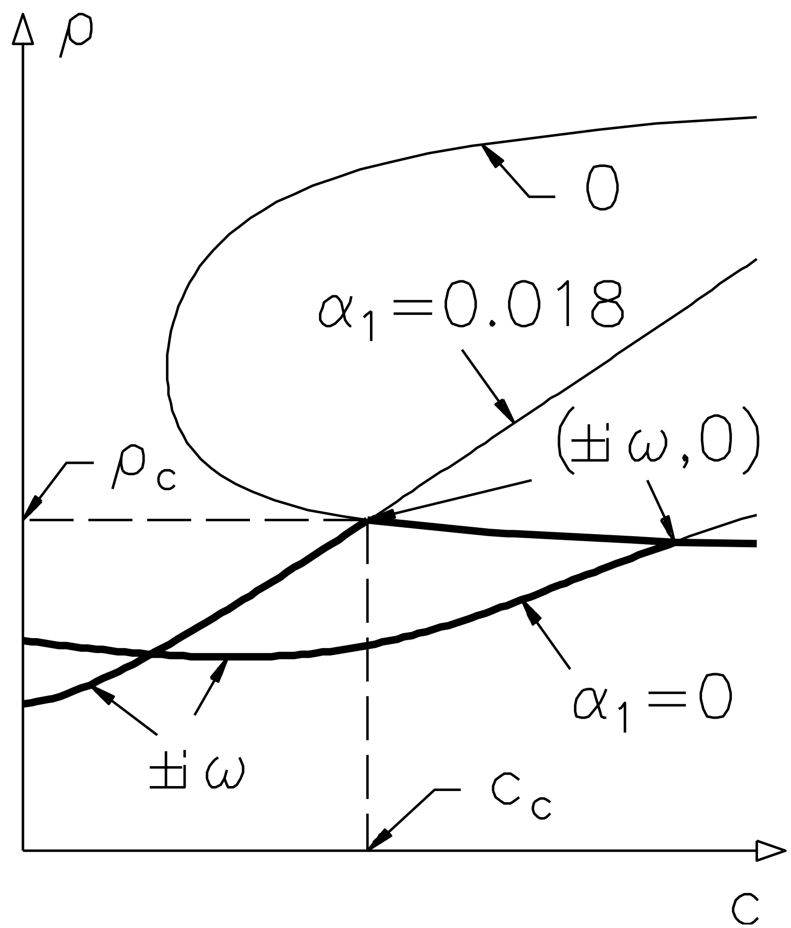
$$\gamma_3(\chi'' + \alpha_3\dot{\chi}'') = 0 \ ,$$

with the boundary and jump conditions

$$
\begin{aligned}
x_i(0) &= 0 \\
x_i'(0) &= 0 \\
\chi(0) &= 0 \\
x_i''(1) + \alpha_1 \dot{x}_i''(1) &= 0 \\
x_i'''(1) + \alpha_1 \dot{x}_i'''(1) &= 0 \\
\gamma_3(\chi'(1) + \alpha_3 \dot{\chi}'(1)) &= 0 \\
x_i'''(\xi_+) + \alpha_1 \dot{x}_i'''(\xi_+) - (x_i'''(\xi_-) + \dot{x}_i'''(\xi_-)) &= -c(x_i(\xi) + \alpha_s \dot{x}_i(\xi)).
\end{aligned}
$$

Since the equation is explicitly depending on the arclength s a numerical solution with the boundary solver BNDSCO (*Oberle, Grimm, Berger 1985*) is given.

Stability boundary in parameter space ($\xi = 0.5$)



O(2)-symmetric coupled Flutter-Divergence instability

At loss of stability we have one purely imaginary pair and one zero root

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with multiplicity two, resulting into a six dimensional system of amplitude equations in real variables and three in complex variables. Up to third order terms they are

$$\dot{z}_1 = i\omega z_1 + \sum_{j+k+l+m+n+p=3} a_{1jklmnp} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m z_2^n \overline{z}_3^p$$

$$\dot{z}_2 = i\omega z_2 + \sum_{j+k+l+m+n+p=3} a_{2jklmnp} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m z_3^n \overline{z}_3^p$$

$$\dot{z}_3 = \sum_{j+k+l+m+n+p=3} a_{3jklmnp} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m z_3^n \overline{z}_3^p.$$

There are 56 nonlinear terms in each of the three equations. But all of them must not be computed in the reduction process. The reason is that the amplitude equations must

1. possess the same symmetry properties as the original system

2. can be strongly simplified by Normal Form Theory (*Arnol'd 1978, Holmes 1981*)

Applying the equivariance test for a rotationally ($\mathbf{O}(2)$) symmetric system according to

Rotation:	$z_1 \rightarrow e^{i\varphi} z_1 \;, \qquad z_2 \rightarrow e^{-i\varphi} z_2 \;, \qquad z_3 \rightarrow e^{i\varphi} z_3$
Reflection	$z_1 \leftrightarrow z_2 \;, \qquad z_3 \leftrightarrow \overline{z}_3$

reduces the number of terms to 18. Subsequent application of Normal Form Theory further reduces the number of terms to 4.

With the two unfolding parameters μ and ν the amplitude equations are

$$
\begin{aligned}
\dot v_1 &= (\mu + i\Omega + A_1|v_1|^2 + A_2|v_2|^2 + A_3|v_3|^2)v_1 + A_4 v_2 v_3^2 \\
\dot v_2 &= (\mu + i\Omega + A_2|v_1|^2 + A_1|v_2|^2 + A_3|v_3|^2)v_2 + A_4 v_1 \overline{v}_3^2 \\
\dot v_3 &= (\quad \nu \qquad + A_5|v_1|^2 + \overline{A}_5|v_2|^2 + A_6|v_3|^2)v_3 + A_7 v_1 \overline{v}_2 \overline{v}_3.
\end{aligned}
$$

In a final step polar coordinates $v_j = r_j e^{i\varphi_j}$ are introduced to obtain

$$
\begin{aligned}
\dot r_1 &= (\mu + c_1 r_1^2 + c_2 r_2^2 + c_3 r_3^2)r_1 + r_2 r_3^2 (c_4 \cos\psi + d_4 \sin\psi) \\
\dot r_2 &= (\mu + c_2 r_1^2 + c_1 r_2^2 + c_3 r_3^2)r_2 + r_1 r_3^2 (c_4 \cos\psi - d_4 \sin\psi) \\
\dot r_3 &= (\nu + c_5(r_1^2 + r_2^2) + c_6 r_3^2 + c_7 r_1 r_2 \cos\psi)r_3 \\
\dot\psi &= (d_1 - d_2 - 2d_5)(r_1^2 - r_2^2) + d_4 r_3^2 \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)\cos\psi - \\
&\quad - c_4 r_3^2 \left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\sin\psi - 2c_7 r_1 r_2 \sin\psi.
\end{aligned}
$$

By introducing $\psi = \varphi_1 - \varphi_2 - 2\varphi_3$, the three differential equations for the phases $\varphi_1, \varphi_2, \varphi_3$ could be combined into one equation for ψ.

Classification of the stationary solutions

Nr.	Orbit repr.	Σ	$\Sigma_x = \mathrm{Fix}(\Sigma)$	$\dim \Sigma_x$	
(0)	$(0,0,0)$	$\mathbf{O}(2) \times \mathbf{S}^1$	$(0,0,0)$	0	TS, straight hanging tube
(1)	$(0,0,b)$	$\mathbf{Z}_2(\kappa) \times \mathbf{S}^1$	$(0,0,x)$	1	SB, planar buckled tube
(2)	$(a,a,0)$	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c$	$(z,z,0)$	2	SW, planar oscillation about TS
(3)	$(a,0,0)$	$\widetilde{\mathbf{SO}}(2)$	$(z,0,0)$	2	TW, rotating motion
(4)	(a,a,b)	$\mathbf{Z}_2(\kappa)$	(z,z,x)	3	oscillation in plane of buckl.
(5)	(a,a,ib)	$\mathbf{Z}_2(\kappa\pi,\pi)$	(z,z,iy)	3	oscillation $\perp$ to buckling plane
(6)	$(a,b,0)$	$\mathbf{Z}_2^c$	$(z_1,z_2,0)$	4	modulated rotating motion
(7)	(a,a,w)	$\mathbf{1}$		6	asymmetric oscillation

Tabelle 1: Stationary solutions and their isotropy subgroups (*Golubitsky, Stewart, Schaeffer 1985*)

Physical interpretation of the stationary solutions

(1) $r_1 = r_2 = r_3 = 0$: TS, vertical hanging tube, $(\mathbf{O}(2) \times \mathbf{S}^1)$

(2) $r_1 = r_2 = 0$, $r_3 \neq 0$: SB, statically buckled, $(\mathbf{Z}_2(\kappa) \times \mathbf{S}^1)$

(3) $r_1 = r_2 \neq 0$, $r_3 = 0$: SW, planar oscillation about TS, $(\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_2^c)$

(4) $r_1 = r_2 \neq 0$, $r_3 \neq 0$, $\sin\psi = 0$: SW about SB. There are two distinct solutions:

- (i) $\psi = 0$: oscillation in the plane of buckling, $(\mathbf{Z}_2(\kappa))$
- (ii) $\psi = \pi$: oscillation orthogonal to the plane of buckling, $(\mathbf{Z}_2(\kappa\pi, \pi))$

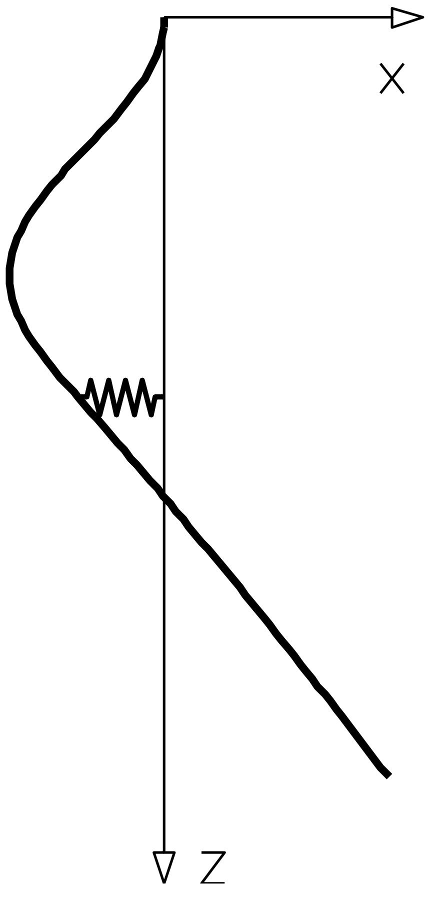
(5)(i) $r_1 \neq 0$, $r_2 = r_3 = 0$: TW, rotating tube, $(\widetilde{\mathbf{SO}}(2))$

(ii) $r_1 = 0$, $r_2 \neq 0$, $r_3 = 0$: TW, rotating in opposite direction, $(\widetilde{\mathbf{SO}}(2))$

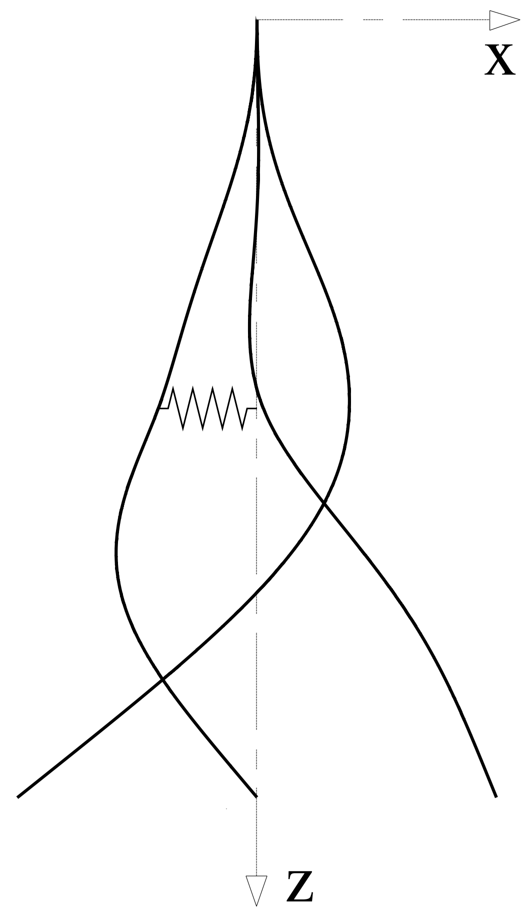
(6) $r_1 \neq 0$, $r_2 \neq 0$, $r_3 = 0$: MW, modulated wave (motion on a torus), $(\mathbf{Z}_2^c)$

(7) $r_1 \neq 0$, $r_2 = O(|r_1 r_3^2|)$, $r_3 \neq 0$: SB with superposed TW.

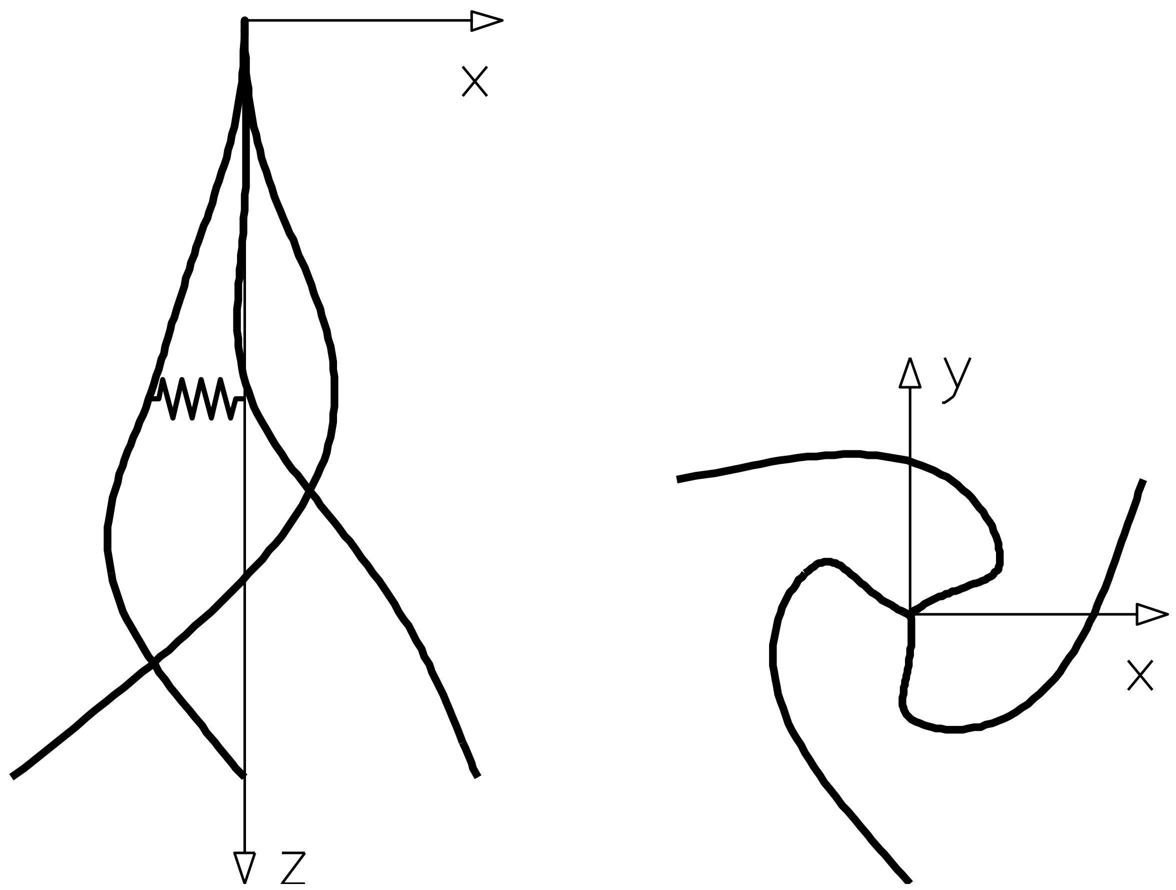
Mode shapes: statical buckled



Mode shapes: standing wave

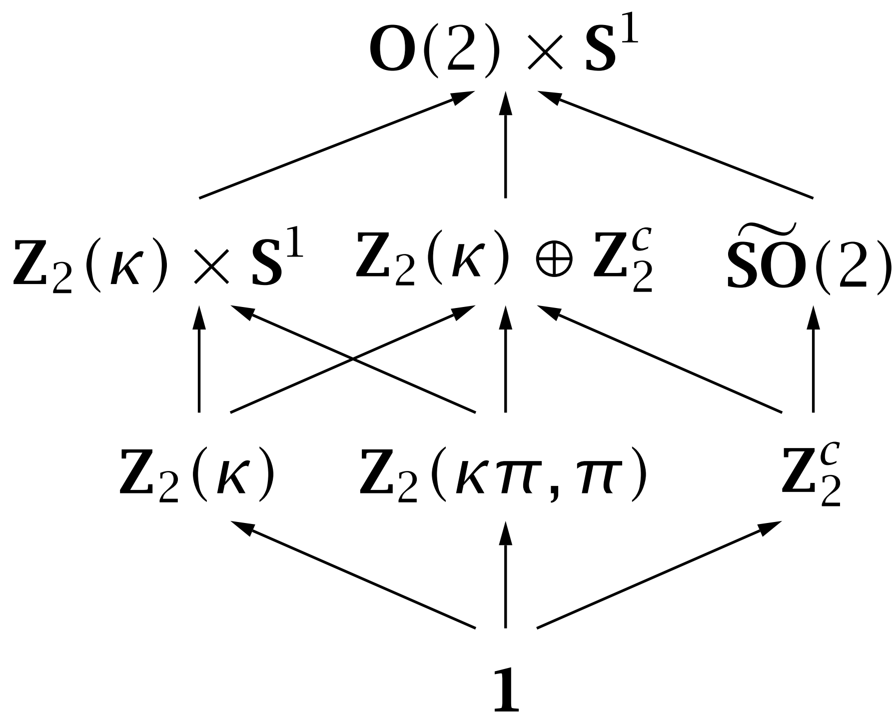


Mode shapes: rotating wave



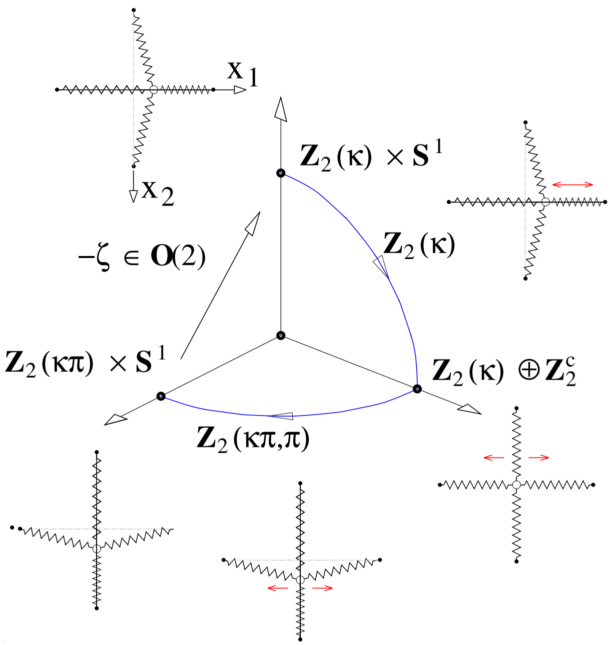
Isotropy-Lattice

The lattice of isotropy subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$ for bifurcation at one zero root and an imaginary pair clearly shows how the symmetries of the different solutions are related to each other



Heteroclinic cycle (*Guckenheimer*, *Golubitsky*, *Krupa*)

The tube moves from the planar buckled state $\mathbf{Z}_2(\kappa) \oplus \mathbf{S}^1$ in the (x_1, x_3) to the planar buckled state $\mathbf{Z}_2(\kappa\pi) \oplus \mathbf{S}^1$ in the (x_2, x_3) plane, which due to the rotational symmetry is the same as the one in the (x_1, x_3) plane.



Global theory

Literature: *Constantin, Foias, Nicolaenko, Temam (1989), Foias, Jolly, Kevrekidis, Sell, Titi (1988), Brunovsky (1993).*

Theoretically important: Inertial manifolds

The importance of the concept of *inertial manifolds* is that it has been proved mathematically that certain dissipative partial differential equations (PDE) possess finite dimensional smooth invariant manifolds called inertial manifolds which contain all the attractors of the PDE.

The inertial manifold attracts exponentially fast all trajectories which start from initial conditions which are not located on the manifold.

The restriction of the PDE to the inertial manifold is an ODE which describes the asymptotic behaviour without error. It can be regarded as the global analog of the normal form of Center Manifold Theory.

Four questions have to be answered in reducing the PDE to an ODE on the inertial manifold:

1. Existence of solutions
2. Compactness of universal attractor
3. Dimension of universal attractor
4. Existence of inertial manifold

Ad 1: Existence of solutions is guaranteed if one can prove that the long time behaviour of the solutions is characterised by a finite dimensional absorbing subset of the phase space. For engineering problems, which include damping, the existence of an absorbing set often can be shown rigorously.

Ad 2: If the phase space of the dynamical system is a Hilbert space and solutions are expanded in an orthogonal basis of this Hilbert space, one would like to show that the higher modes in this basis decay strongly.

Ad 3: Estimating Hausdorff or fractal dimension of the universal attractor is based upon linearizing the system along its trajectories and computing a Lyapunov spectrum. A rough estimate of part of the Lyapunov spectrum can be obtained from looking at the growth rates of n-dimensional volumes in the linearized flow. If there is some n, for which all n-dimensional volumes decrease along the flow, then n is an upper bound for the dimension of the universal attractor.

Ad 4: One can hope that not only the attractors of an infinite dimensional system will be finite, but that there will be a smooth finite dimensional subset that is invariant under the flow and contains the universal attractor. Such a subset is called an *Inertial Manifold*.

The existence of inertial manifolds is a more delicate matter than the existence of universal attractors.

The question that one asks is when a smooth invariant submanifold in a dynamical system will persist under perturbation. The attracting invariant manifold persistent under perturbations must have more extreme Lyapunov exponents in its normal directions than in its tangential directions.

If the partial differential equations being studied have large gaps in their spectra, then these can be used to look for invariant manifolds that lie close to the linear space spanned by the modes whose eigenvalues lie to the right of a gap in the complex plane.

There are many examples for which such spectral gaps exist. For example for a reaction-diffusion equation in one dimension, the eigenvalues of the Laplacian give eigenvalues that decay in magnitude like $-n^2$ and this leads to the existence of the appropriate gap conditions.

The solution may be expanded in an orthonormal basis of a Hilbert space assuming that the higher modes decay fast. As in Center Manifold theory the concept of active modes

$$u(x,t) = q_1(t)w_1(x) + \cdots + q_N(t)w_m(x) + \cdots$$

is introduced. The modes $w_1(x), \cdots, w_m(x)$ are called active if they describe the asymptotic behavior for $t \to \infty$. If we express

$$v(x,t) = q_1(t)w_1(x) + \cdots + q_m(t)w_m(x)$$

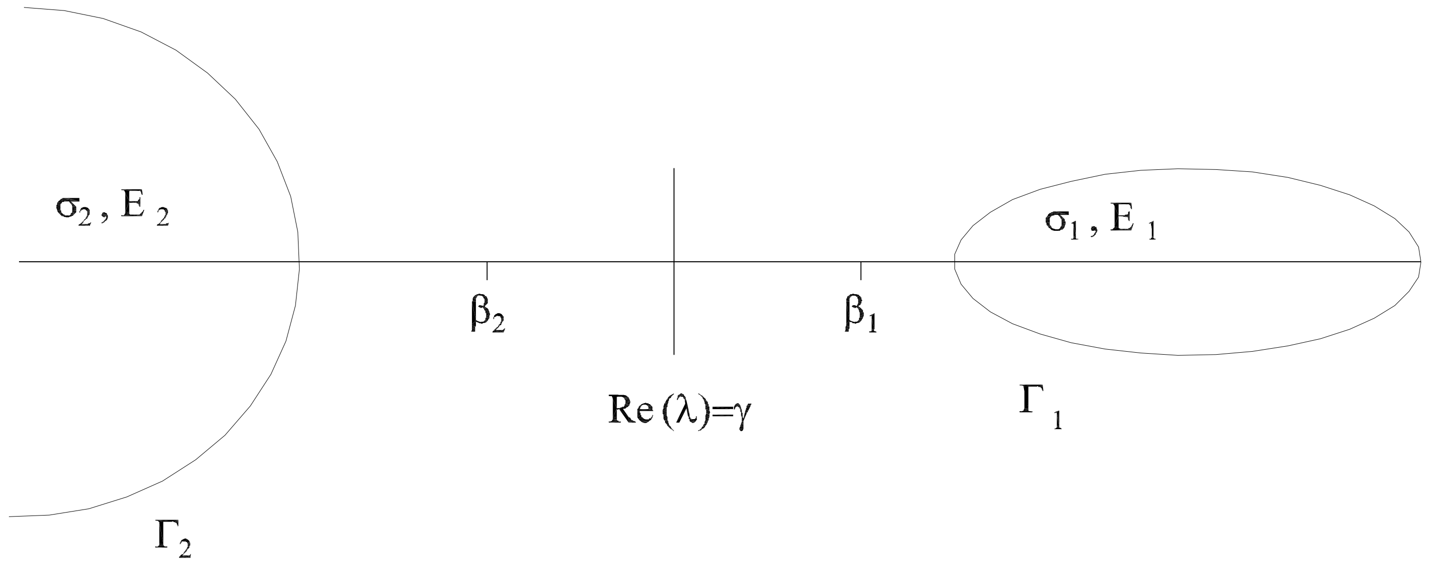
this results in

$$\int_\Omega |u(x,t) - v(x,t)|^2 dx \to 0 \quad \text{for} \quad t \to \infty.$$

The question of the number of active modes and the existence of a smooth invariant manifold called Inertial Manifold which contains the attractor depends on a "gap condition" in the spectrum of the linear operator.

Meaning of the gap condition

We ask the question: Does a smooth invariant manifold of a dynamical system persist under perturbation?



Consider an operator $A : E \to E$. Assume that the spectrum of A, $\sigma(A)$, admits a decomposition

$$\sigma(A) = \sigma_1 \cup \sigma_2$$

$$Re\lambda > \beta_1 \quad \text{for} \quad \lambda \in \sigma_1, \qquad Re\lambda < \beta_2 \quad \text{for} \quad \lambda \in \sigma_2, \qquad \text{and} \quad \beta_1 > \beta_2. \tag{7}$$

Let $E = E_1 \oplus E_2$ be the decomposition of the space E into invariant subspaces corresponding to the decomposition of $\sigma(A)$, $PA = A_1 := A\mid_{E_1}$, $QA = A_2 := A\mid_{E_2}$.

Question: Given $\gamma \in (\beta_2, \beta_1)$, how large can the norm of a bounded operator B be in order that the gap in the spectrum of the perturbed operator $A + B$ persists (i.e., it stays away from the line $Re\lambda = \gamma$)?

In other words, any $\lambda \in C$ with $Re\lambda = \gamma$ should not be in the spectrum of $A + B$: Hence $A + B - \lambda I$ should be invertible. From the identity

$$A + B - \lambda I = (A - \lambda I)(I + (A - \lambda I)^{-1}B)$$

follows, $\lambda \notin Sp(A + B)$ provided

$$\mid (A - \lambda I^{-1}) \mid\mid B \mid < 1.$$

From the Cauchy formula for the functions of the operator A (holds e.g. if A is bounded or unbounded sectorial) we get

$$
\begin{aligned}
&| (A-\lambda I)^{-1} |= \\
&=| \frac{1}{2\pi i}\int_{\Gamma_1}(\mu-\lambda)^{-1}(A_1-\mu I)^{-1}d\mu \mid + \mid \frac{1}{2\pi i}\int_{\Gamma_2}(\mu-\lambda)^{-1}(A_2-\mu I)^{-1}d\mu \mid \\
&=| \frac{1}{2\pi i}\int_{\Gamma_1}\int_{-\infty}^{0} e^{(\mu-\lambda)t}dt(A_1-\mu I)^{-1}d\mu \mid + \mid \frac{1}{2\pi i}\int_{\Gamma_2}\int_{0}^{\infty} e^{(\mu-\lambda)t}dt(A_2-\mu I)^{-1}d\mu \mid \\
&=| \int_{-\infty}^{0} e^{-\lambda t}\frac{1}{2\pi i}\int_{\Gamma_1} e^{\mu t}(A_1-\mu I)^{-1}d\mu dt \mid + \mid \int_{0}^{\infty} e^{-\lambda t}\frac{1}{2\pi i}\int_{\Gamma_2} e^{\mu t}(A_2-\mu I)^{-1}d\mu dt \mid \\
&= \int_{-\infty}^{0} e^{-\lambda t} \mid e^{A_1 t} \mid dt + \int_{0}^{\infty} e^{-\lambda t} \mid e^{A_2 t} \mid dt \\
&\leq M[\int_{-\infty}^{0} e^{(-\gamma+\beta_1)t}dt + \int_{0}^{\infty} e^{(-\gamma+\beta_2)t}dt] \\
&= M[\frac{1}{\beta_1-\gamma}+\frac{1}{\gamma-\beta_2}]
\end{aligned}
$$

where Γ_1, Γ_2 are (possibly unbounded) positively oriented curves encircling σ_1, σ_2.

Thus, $\sigma(A+B) \cap \{Re\lambda = \gamma\} = \emptyset$ provided that

$$\mid B \mid M[\frac{1}{\beta_1 - \gamma} + \frac{1}{\gamma - \beta_2}] < 1$$

i.e. the norm of B is small compared to the gap in the spectrum. In other words: γ devides the spectrum.

For several important classes of operators generating semigroups A_1 turns out to be bounded and (7) implies that $e^{A_1 t}$, $e^{A_2 t}$ satisfy the estimates:

$$\mid e^{A_1 t} \mid \leq M e^{\beta_1 t} \quad \text{for} \quad t \leq 0, \qquad \Leftrightarrow \quad \mid e^{A_1 t} \mid \geq \frac{1}{M} e^{\beta_1 t} \quad \text{for} \quad t \geq 0 \tag{8}$$

$$\mid e^{A_2 t} \mid \leq M e^{\beta_2 t} \quad \text{for} \quad t \geq 0 \tag{9}$$

for some $M > 0$. If A is bounded this is always the case.

The estimates (8), (9) imply that E splits into two transversal subspaces E_1, E_2 such that the exponential rate of increase of the trajectories of the differential equation

$$\dot{u} = Au$$

is strictly higher than γ in E_1 and strictly lower than γ in E_2 (for $\gamma \leq 0$ "increase" may still be negative). Coordinates exist in which the equation decouples

$$\dot{u_1} = A_1 u_1$$
$$\dot{u_2} = A_2 u_2.$$

For a linear perturbation such a splitting still exists, provided that the norm of the perturbation is sufficiently small compared to the gap in the spectrum of A.

Invariant manifold theorems extend this result to a linear differential equation being perturbed by a nonlinear function

$$\dot{u} = Au + f(u)$$

which is sufficiently small expressed by the Lipschitz constant.

Inertial Manifold for Nonlinear Autonomous System

$$\dot{u} = Au + f(u) \tag{10}$$

H1 $\sigma(A) = \sigma_1 \cup \sigma_2$ with $Re\lambda_2 < \beta_2 < \beta_1 < Re\lambda_1$ for all $\lambda_1 \in \sigma_1$, $\lambda_2 \in \sigma_2$

H2 f satisfies: $f(0) = 0$ and uniformly globally Lipschitz continuous (constant L) in u,

$$\mid f(u) - f(\tilde{u}) \mid \leq L \mid u - \tilde{u} \mid$$

Then we can write (10) in the form

$$\dot{u}_1 = A_1 u_1 + f_1(u_1, u_2)$$
$$\dot{u_2} = A_2 u_2 + f_2(u_1, u_2)$$

Fix $\gamma \in (\beta_1, \beta_2)$ and denote

$$\kappa := M[\frac{\mid P \mid}{(\beta_1 - \gamma)} + \frac{\mid Q \mid}{(\gamma - \beta_2)}] \tag{11}$$

Assume that hypotheses $(H1)$ and $(H2)$ hold and that

$$L\kappa < 1$$

Then, there exists a function $W_1 : E_1 \times \mathbb{R} \to E_2$, Lipschitz continuous in u with constant

$$\Lambda := \frac{ML\kappa}{1 - L\kappa}$$

In the considered autonomous case $u_2 = W_1(u_1)$ is an invariant manifold.

If $\gamma < 0$ and $\Lambda < 1$ the manifold W_1 has the *tracking property*: for each $u \in U$ there is a unique point $y \in W_1$ such that $\mid g^t(u) - g^t(y) \mid \to 0$ for $t \to \infty$.

The restriction of the flow to the manifold W_1 is governed by the differential equation

$$\dot{u}_1 = A_1 u_1 + f(u_1, W_1(u_1))$$

called the *reduction* of (10) to W_1. The tracking property has as its immediate consequence that due to $\gamma < 0$ and $\Lambda < 1$ all attractors lie in W_1.

Applications

The essential condition for the existence of an inertial manifold

$$L\kappa < 1$$

can be satisfied in two ways: either the Lipschitz constant L of f or the constant κ defined by (11) is small. The latter option means that the gap must be large enough.

1) Laplacian $A = \Delta$: one gets the following estimates for the magnitude of the eigenvalues:

$$\lambda_j \sim j^{\frac{2}{d}}$$

where d is the dimension of the space.

Hence for

$d = 1 \quad \lambda_j \sim j^2, \quad \lambda_{j+1} - \lambda_j \sim j$ growing gap exists for growing j

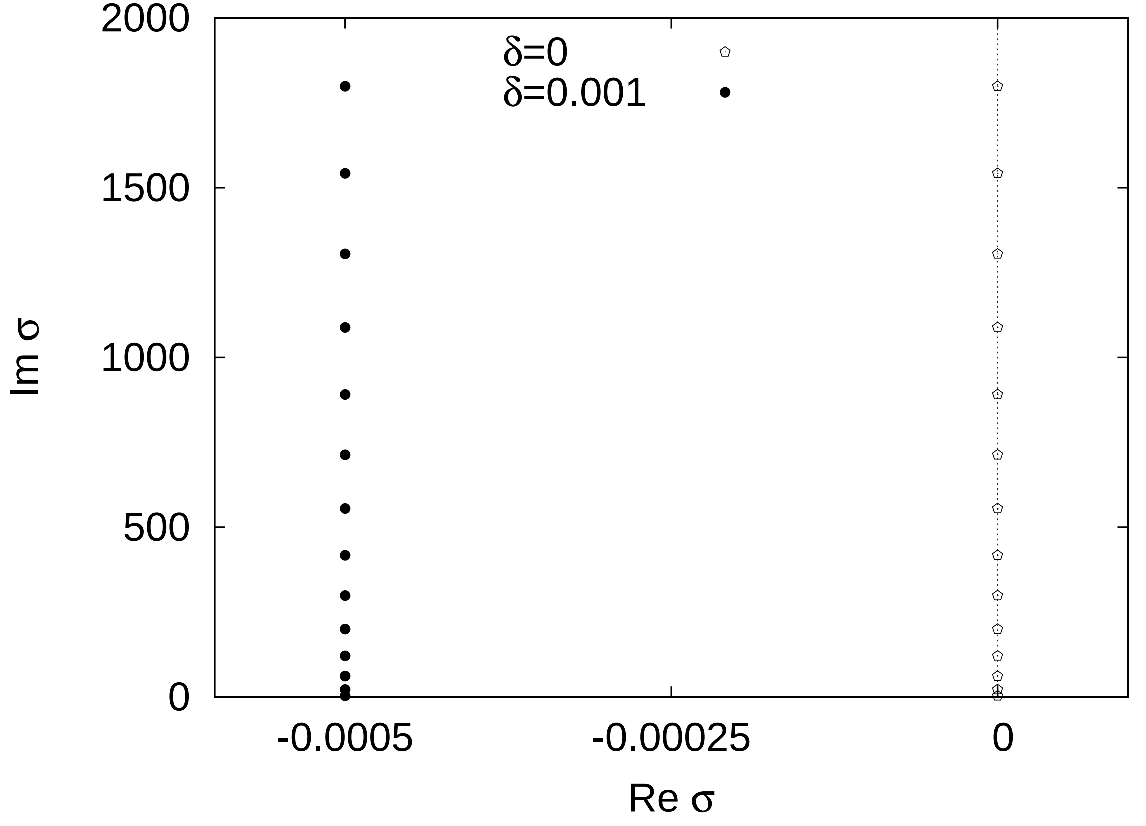
and for

$d = 2 \quad \lambda_j \sim j, \quad \lambda_{j+1} - \lambda_j \sim 1$ growing gap does not exist.

2) Clamped-free beam with external damping

$$\ddot{u} + \delta \dot{u} + u^{IV} = 0$$

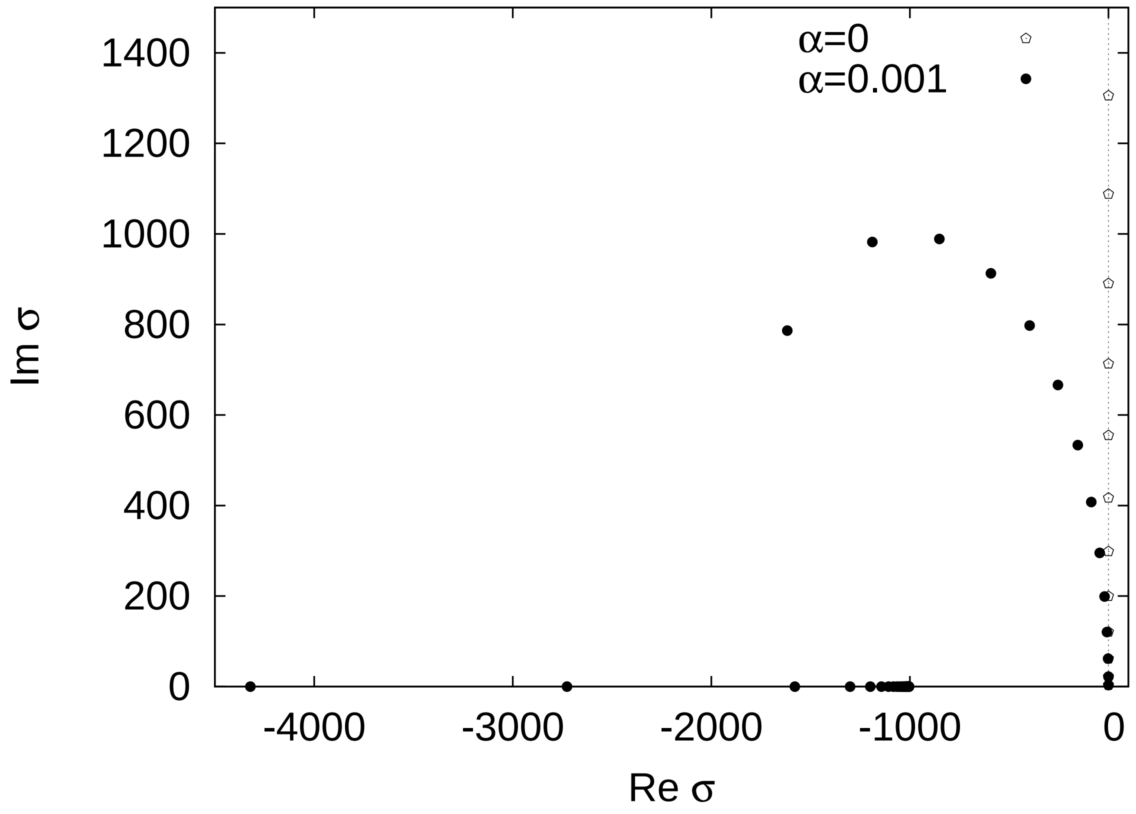
$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0$$



3) Clamped-free beam with internal damping

$$\ddot{u} + \alpha_1 \dot{u}^{IV} + u^{IV} = 0$$

$$u(0) = 0, \quad u'(0) = 0, \quad u''(1) + \alpha_1 \dot{u}''(1) = 0, \quad u'''(1) + \alpha_1 \dot{u}'''(1) = 0$$



Nonlinear Galerkin (Approximate Inertial Manifold)

$$\dot{\boldsymbol{u}} = \boldsymbol{A}(\lambda)\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{u}, \lambda),$$

and make the finite dimensional ansatz:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \underbrace{\boldsymbol{u}_c}_{R^m} + \underbrace{\boldsymbol{u}_s}_{R^{n-m}}, \qquad n \ldots \text{shape functions}$$

An iterative scheme to calculate an approximation of the "inertial manifold" is set up:

1. Discretize the original infinite dimensional system by an n-term Galerkin ansatz

2. Designate m essential variables by $\boldsymbol{p} \subset \mathbb{R}^m$ and noncritical variables by $\boldsymbol{q} \subset \mathbb{R}^{n-m}$. Select the essential modes corresponding to the distribution of the eigenvalues in the complex plane. Check for a "gap condition" separating the other eigenvalues from the critical ones.

3. Rewrite the n-th order system in the form

$$\frac{d\boldsymbol{p}}{dt} = \mathbf{A}_1\boldsymbol{p} + P\mathbf{g}(\boldsymbol{p} + \boldsymbol{q}) \tag{12}$$

$$\frac{d\boldsymbol{q}}{dt} = \mathbf{A}_2\boldsymbol{q} + Q\mathbf{g}(\boldsymbol{p} + \boldsymbol{q}) \tag{13}$$

where $\mathbf{A}_1 = P\mathbf{A}$ and $\mathbf{A}_2 = Q\mathbf{A}$ and $P : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $Q = I - P$ are proper projections.

4. Write the reduced system

$$\frac{d\boldsymbol{p}}{dt} = \mathbf{A}_1\boldsymbol{p} + P\mathbf{g}(\boldsymbol{p} + \boldsymbol{\phi}_a(\boldsymbol{q})) \ ,$$

where $\boldsymbol{\phi}_a$ is an approximation of the "inertial manifold".

5. Create an iterative scheme. Set $d\boldsymbol{q}/dt = 0$ to obtain the mapping

$$\boldsymbol{q} = -\mathbf{A}_2^{-1} Q\mathbf{g}(\boldsymbol{p} + \boldsymbol{q}) \ .$$

The approximate inertial manifold $\boldsymbol{\phi}_a$ is given as its fixed point. Approximate it by $\boldsymbol{\phi}_i$. Starting with $\boldsymbol{\phi}_0 \equiv 0$ (linear approximation, flat Galerkin) one obtains

$$\begin{aligned} \boldsymbol{\phi}_1 &= -\mathbf{A}_2^{-1} Q\mathbf{F}(\boldsymbol{p}) \\ \boldsymbol{\phi}_2 &= -\mathbf{A}_2^{-1} Q\mathbf{F}(\boldsymbol{p} + \boldsymbol{\phi}_1(\boldsymbol{p})) \ \ldots . \ . \end{aligned}$$

6. Due to the neglection of $\dot{\boldsymbol{q}}$ it is questionable to go to high approximations.

7. Post-processed Galerkin method (*E. Titi*)

   In the approximate inertial manifold approach (12) is replaced by

$$\frac{d\boldsymbol{p}_a}{dt} = P\mathbf{A}\boldsymbol{p}_a + P\mathbf{F}(\boldsymbol{p}_a + \boldsymbol{\phi}_a(\boldsymbol{p}_a)) \ , \tag{14}$$

where $\boldsymbol{\phi}_a$ is an approximation of the inertial manifold.

To solve (14), first, at each step $\boldsymbol{\phi}_a$ must be computed and, second, the integration of (14), is also more costly than the integration of the flat Galerkin equation

$$\frac{d\boldsymbol{p}_f}{dt} = P\mathbf{A}\boldsymbol{p}_f + P\mathbf{F}(\boldsymbol{p}_f) \ , \tag{15}$$

where $\boldsymbol{\phi}_a = \mathbf{0}$. However, $\boldsymbol{p}_f$ calculated from (15) will be in general less accurate then $\boldsymbol{p}_a$ calculated from (14).

In *Garcia, Novo, Titi (1998)* the post-processed Galerkin method is used to calculate $\boldsymbol{p}_f$ from (15) and only if output is required at time τ to lift it up to $\boldsymbol{\phi}_a$ yielding

$$\boldsymbol{u}_{\mathsf{pp}}(\tau) \approx \boldsymbol{p}_f(\tau) + \boldsymbol{\phi}_a(\boldsymbol{p}_f(\tau)).$$

whereas $\boldsymbol{u}_{\mathsf{aim}}(\tau)$ computed from the nonlinear Galerkin equation (14) is

$$\boldsymbol{u}_{\mathsf{aim}}(\tau) \approx \boldsymbol{p}_a(\tau) + \boldsymbol{\phi}_a(\boldsymbol{p}_a(\tau)).$$

Computationally a great reduction is achieved since lifting is done only when output is required and not at every time step in the integration.

Comparison of CM and AIM

We consider two losses of stability of an equilibrium: (1) divergent and (2) flutter.

Example 1: Divergent bifurcation We consider

$$\begin{aligned} \dot{x} &= ax^2 + bxy + fx^3 \\ \dot{y} &= -y + cx^2 + dxy + ex^3 \ . \end{aligned}$$

we make an ansatz for the center manifold in the form

$$y = H(x) = h_2 x^2 + h_3 x^3 + \ldots \ .$$

Up to third order one obtains

$$(2h_2 x)(ax^2) + h_2 x^2 + h_3 x^3 - cx^2 - dxh_2 x^2 - ex^3 = 0 \ .$$

Vanishing of coefficients of second and third order terms

$$\begin{aligned} h_2 - c &= 0 \\ 2h_2 a + h_3 - dh_2 - e &= 0 \ , \end{aligned}$$

resulting in the coefficients

$$\begin{aligned} h_2 &= c \\ h_3 &= e + cd - 2ac \ . \end{aligned}$$

For the AIM we obtain

$$\begin{aligned} \phi_1 &= cx^2 \\ \phi_2 &= cx^2 + dcx^3 + ex^3 \ . \end{aligned}$$

Comparison with the CM-result shows that the quadratic term is the same, but that the cubic is already different. Hence AIM gives a correct result up to third order terms in the amplitude equation.

Example 2: Hopf bifurcation We study the set of equations

$$\begin{aligned}\dot{z} &= i\omega z + a|z|^2 + bzu + \dots \\ \dot{u} &= -\mu u + c_{20}z^2 + c_{11}|z|^2 + c_{02}\bar{z}^2 + duz + \dots \ .\end{aligned}$$

For CM reduction we make an ansatz

$$u = H(z) = h_{20}z^2 + h_{11}|z|^2 + h_{02}\bar{z}^2 + \dots \ .$$

Inserting we obtain

$$2h_{20}z\dot{z}+h_{11}(z\dot{\bar{z}}+\dot{z}\bar{z})+2h_{02}\bar{z}\dot{\bar{z}} = -\mu(h_{20}z^2+h_{11}|z|^2+h_{02}\bar{z}^2)+c_{20}z^2+c_{11}|z|^2+c_{02}\bar{z}^2+\text{h.o.t.}$$

Now we substitute in this expression for $\dot{z} = i\omega z$ and $\dot{\bar{z}} = -i\omega\bar{z}$ to obtain

$$2i\omega h_{20}z^2 - 2i\omega h_{02}\bar{z}^2 = (c_{20} - \mu h_{20})z^2 + (c_{11} - \mu h_{11})|z|^2 + (c_{02} - \mu h_{02})\bar{z}^2 + \text{h.o.t.}$$

For the lowest order (quadratic) terms we obtain

$$h_{20} = \frac{c_{20}}{\mu + 2i\omega}, \qquad h_{11} = \frac{c_{11}}{\mu}, \qquad h_{02} = \frac{c_{02}}{\mu - 2i\omega} \ . \tag{16}$$

The lowest order terms of the AIM are

$$h_{20} = \frac{c_{20}}{\mu} \ , \qquad h_{11} = \frac{c_{11}}{\mu} \qquad h_{02} = \frac{c_{02}}{\mu} \ .$$

They are distinct from the CM values.

Hence for a Hopf bifurcation (flutter instability) the neglection of the dynamic term in the passive set of equations can already result in incorrect amplitude (bifurcation) equations of lowest (third) order.

However if $\mid \mu \mid \gg \omega$ the AIM is a good approximation.

If the nonlinearities start with cubic terms then instead of the denominator $\mu + 2i\omega$ in (16) a term $\mu + 3i\omega$ appears, resulting in even a larger difference.

Foias, Jolly, Kevrekidis, Sell and Titi 1988

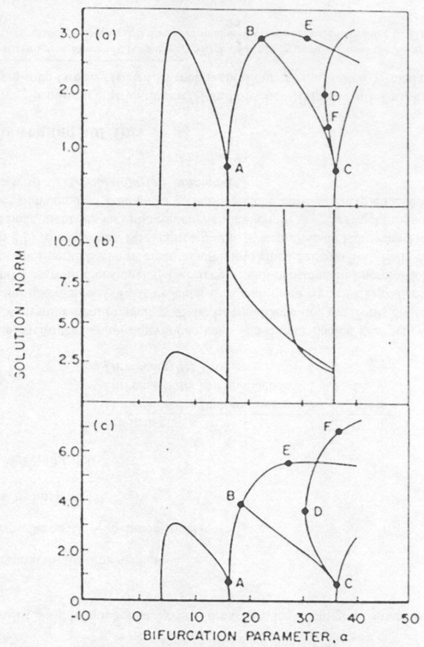


Fig. 2. Steady state bifurcation diagrams (euclidean norm of the solution versus α) for the KS equation obtained through traditional Galerkin approximations for (a) $n=16$ and (b) $n=3$, and (c) through an Euler–Galerkin approximation ($n=3$, $m=6$, $\tau=0.001$).

$$u_t + 4u_{xxxx} + \alpha u_{xx} + \frac{1}{2}\alpha u_x^2 = 0$$

$$u(x,t) = u(x+2\pi, t)$$

Karhunen Loeve method

The Karhunen Loeve method or *Proper Orthogonal Decomposition* method (Sirovich 1989, Holmes, Lumley, Berkooz 1996) allows to generate an optimal set of basis functions $(\phi_1, \dots, \phi_m)$, based on second-order statistics.

By optimal we mean that compared to any other choice of basis functions for the same number m a better approximation is achieved.

However, data from experiments or simulation are necessary to apply this method.

Basic Idea of the Karhunen Loeve method (Lumley 1971)

Having some experimental or simulation data available, the basic idea is to find a deterministic function f which in some statistical sense has the structure typical of the members u of the ensemble obtained from experiments or simulation.

Assume that u is a vector in a function space then f should be as nearly parallel to u in a statistical sense as possible.

We assume that the projection of u on f given by (u, f) is defined. The task is to maximize (u, f)

Obviously one can increase the value of (u, f) simply by increasing the magnitude of f without changing its form. Hence, one must normalize by the length of the vector f

$$\frac{(u, f)}{(f, f)^{\frac{1}{2}}}.$$

We restrict to a Hilbert space of functions f where (f, f) exists.

Since the f which maximizes (u, f) is just u one has to maximize in some average sense.

If the mean value of u is zero one would obtain $E\{(u, f)\}) = 0$. Hence instead of $\frac{(u,f)}{(f,f)^{\frac{1}{2}}}$ the quadratic expression

$$\frac{E\{(u, f)(u, f)\}}{(f, f)} = \lambda \geq 0.$$

will be maximized.

The solution as stated would yield only the best approximation to the esemble by a single function. But the other critical points of this functional are also physically

significant. They belong to a set of functions providing the desired basis.
The problem can be stated in the calculus of variations: Extremize $E\{(u,f)(u,f)\}$ subject to the constraint $(f,f)=1$. The corresponding functional for this constraint variational problem is:

$$J[f] = E\{(u,f)(u,f)\} - \lambda((f,f)-1).$$

Requiring that the functional derivative of J is zero, results in the eigenvalue problem

$$\int_D E\{u(x)u(x')\}f(x')dx' = \lambda f(x).$$

whose kernel is the averaged autocorrelation function

$$E\{u(x)u(x')\} = R(x,x')$$

Its solution supplies, first, the set of optimal eigenmodes f_i and, second, the corresponding eigenvalues λ_i which measure the energy content carried by the various modes.

By optimal eigenfunctions or eigenvectors a system is understood which approximates the data in such a way that the eigenvectors are parallel to the axis of the energy ellipsoid. Here also an analogy to the inertia ellipsoid is given.

Karhunen-Loeve method applied to Galerkin approximation

We assume that an ensemble $\{\boldsymbol{u}_i\}_{i=1}^{p}$ of p pattern vectors is obtained from numerical simulation, where each $\boldsymbol{u}_i \in V = \mathbb{R}^n$. Let B be a basis $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ such that any $\boldsymbol{u}_i$ can be represented by

$$\boldsymbol{u}_i = \sum_{j=1}^{n} b_{ij} \boldsymbol{v}_j \ .$$

We define now the ensemble average of a set of p pattern vectors by

$$E\{\boldsymbol{u}\} = \langle \boldsymbol{u} \rangle = \frac{1}{p} \sum_{i=1}^{p} \boldsymbol{u}_i \ .$$

Often a time continuous quantity $\boldsymbol{u}(t)$ is either measured or computed and stored at discrete time steps t_i. For equidistant time steps

$$\boldsymbol{u}(t_i) = \sum_{j=1}^{n} b_j(t_i) \boldsymbol{v}_j \ .$$

The averaging process

$$\langle \boldsymbol{u} \rangle = \lim_{t \to \infty} \frac{1}{T} \int_0^T \boldsymbol{u}(t) dt$$

yields, replacing the integral by a finite sum,

$$\langle \boldsymbol{u} \rangle = \langle \boldsymbol{u}(t_i) \rangle = \frac{1}{p \Delta t} \sum_{i=1}^{p} \left(\sum_{j=1}^{n} b_j(t_i) \boldsymbol{v}_j \right) \Delta t \ .$$

Since Δt cancels out we again end up with the expression given above if we use $\boldsymbol{u}_i = \boldsymbol{u}(t_i)$.

The aim is to represent $\boldsymbol{u}_i$ in a new basis $\widetilde{B} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)$ in the form

$$\boldsymbol{u}_i = \sum_{j=1}^{n} a_{ij} \boldsymbol{\phi}_j .$$

such that the ensemble average error e_{ms} of the truncated expansion

$$\boldsymbol{u}_i \approx \boldsymbol{u}_{im} = \sum_{j=1}^{m} a_{ij} \boldsymbol{\phi}_j$$

will have minimal error.
The general theory requires the minimization of the quadratic expression

$$e_{ms} = \langle \| \boldsymbol{u} - \boldsymbol{u}_m \|^2 \rangle.$$

The task to be performed is an optimal decomposition of the vectors

$$\boldsymbol{u}_i = \boldsymbol{u}_{im} + \boldsymbol{e}_i$$

where $\boldsymbol{u}_{im} \in W_m$, $\boldsymbol{e}_i \in W_m^{\perp}$ with $V = W_m \oplus W_m^{\perp}$.
Inserting results in

$$\boldsymbol{e}_i = \boldsymbol{u}_i - \boldsymbol{u}_{im} = \sum_{j=m+1}^{n} a_{ij}\boldsymbol{\phi}_j.$$

Forming the quadratic minimization problem for e_{ms} we have

$$e_{ms} = \left\langle \left(\sum_{j=m+1}^{n} a_j\phi_j, \sum_{k=m+1}^{n} a_k\phi_k \right) \right\rangle = \left\langle \sum_{j,k} a_j a_k (\phi_j, \phi_k) \right\rangle$$

$$= \left\langle \sum_{j=m+1}^{n} a_j^2 \right\rangle = \left\langle \sum_{j=m+1}^{n} (u, \phi_j)^2 \right\rangle$$

We note that

$$(\phi, u)^2 = (u^T\phi)^T(u^T\phi) = \phi^T u u^T \phi = (\phi, uu^T\phi)$$

and with

$$C = \langle uu^T \rangle$$

we have

$$e_{ms} = \left\langle \sum_{j=m+1}^{n} (\phi_j, uu^T \phi_j) \right\rangle = \sum_{j=m+1}^{n} (\phi_j, \langle uu^T \rangle \phi_j) = \sum_{j=m+1}^{n} (\phi_j, C\phi_j)$$

To determine the minimum of e_{ms} we use the Lagrange multiplier method. From

$$L(\phi) = \sum_{j=m+1}^{n} (\phi_j, C\phi_j) - \sum_{j=m+1}^{n} \lambda_j \left((\phi_j, \phi_j) - 1 \right)$$

follows by setting the Frechet derivative to zero

$$(C - \lambda_j I)\phi^j = 0$$

That is, the ϕ_j are the solution of the eigenvalue problem

$$C\phi_j = \lambda\phi_j$$

where C is the ensemble averaged covariance matrix

$$C = \langle uu^T \rangle$$

Some properties of the Karhunen–Loeve method:

1. If $\langle \boldsymbol{u} \rangle = 0$ then $\langle a_j \rangle = 0$,

   $\langle a_j \rangle = \left\langle (\boldsymbol{u}, \boldsymbol{\phi}_j) \right\rangle = (\langle \boldsymbol{u} \rangle, \boldsymbol{\phi}_j) = \left\langle \mathbf{0}, \boldsymbol{\phi}_j \right\rangle = \mathbf{0}$.

2. The optimal eigenvectors diagonalize the covariance matrix.

3. The eigenvalues of $\mathbf{C}$ are non-negative

$$\lambda_j = \left\langle a_j^2 \right\rangle \geq 0 \qquad j = 1, \ldots, n.$$

   If the basis $\{\boldsymbol{q}^j\}$ is ordered in accordance to

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

   then

$$\underline{e_{ms}} = \sum_{j=m+1}^{n} (\boldsymbol{\phi}^j, \mathbf{C}\boldsymbol{\phi}^j) = \sum_{i=m+1}^{n} (\boldsymbol{\phi}^j, \lambda_j \boldsymbol{\phi}^j) = \sum_{j=m+1}^{n} \lambda_j \ ,$$

will be a *minimum.*

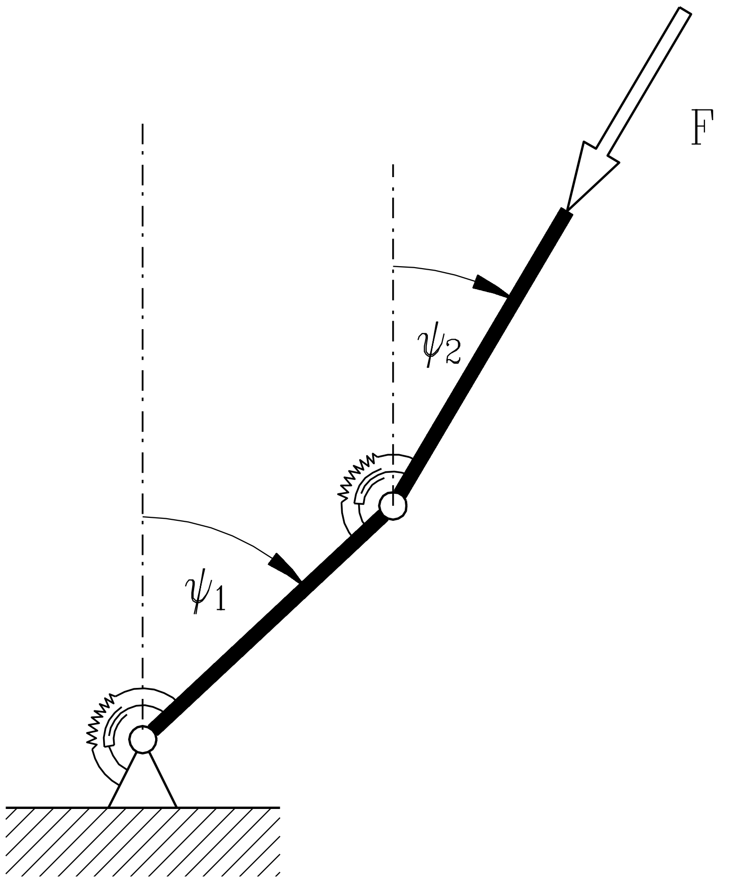
4. Since $a_k = (\boldsymbol{u}, \boldsymbol{\phi}_k)$ we obtain

$$\begin{aligned}\langle a_k a_j \rangle &= \langle (\boldsymbol{u}, \boldsymbol{\phi}_k)(\boldsymbol{u}, \boldsymbol{\phi}_j) \rangle = \langle (\boldsymbol{\phi}_k, \boldsymbol{u}\boldsymbol{u}^T \boldsymbol{\phi}_j) \rangle = \big(\boldsymbol{\phi}_k, \langle \boldsymbol{u}\boldsymbol{u}^T \rangle \boldsymbol{\phi}_j\big) \\ &= \big(\boldsymbol{\phi}_k, \mathbf{C}\boldsymbol{\phi}_j\big) = \big(\boldsymbol{\phi}_k, \lambda_j \boldsymbol{\phi}_j\big) = \lambda_j \delta_{kj} \ .\end{aligned}$$

Hence the eigenvalue $\lambda_j = \langle a_j a_j \rangle$ corresponds to the statistical variance or can be considered to be a measure of the "kinetic energy" in the j-th eigenmode.

If the eigenvalues are normalized as probabilities the percentual energy content of an approximation can be estimated.

Example: Double pendulum with follower force



The equations of motion are

$$\begin{pmatrix} 4 & \frac{3}{2}\cos(\psi_1-\psi_2) \\ \frac{3}{2}\cos(\psi_1-\psi_2) & 1 \end{pmatrix} \begin{pmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{pmatrix} = \begin{matrix} -\frac{3}{2}\dot{\psi}_2^2\sin(\psi_1-\psi_2) \\ \frac{3}{2}\dot{\psi}_1^2\sin(\psi_1-\psi_2) \end{matrix}$$

$$-\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} - \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_2 \\ -\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{matrix} F\sin(\psi_1-\psi_2) \\ 0 \end{matrix} \quad .$$

They are already divided by $\frac{m\ell^2}{3}$. Linearization about $\psi_1 = \psi_2 = 0$ yields

$$\begin{pmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{pmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix}$$

$$+ \begin{pmatrix} \gamma_1+\gamma_2-F & -\gamma_2+F \\ -\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

For $\gamma_1 = \gamma_2 = k_1 = 1.0$ and $k_2 = 0.6$ the critical value $F = F_c$ for loss of stability is $F_c = 2.196$.

At $F = F_c$ a stable (supercritical) limit cycle bifurcates from the equilibrium. The nonlinear equations of motion are simulated for $F = 3.0 > F_c$ (in the domain of the stable limit cycle)

With $\boldsymbol{u} = (\psi_1, \psi_2)^T$ and the stepsize $\Delta t = 0.25\,\text{sec}$ $p = 2000$ samples from the simulation data are taken. the covariance matrix

These covariance matrices for minimizing the error in position (p), velocity (v) and acceleration (a) are

$$\mathbf{C}_p = \begin{pmatrix} .2952 & .5587 \\ .5587 & 1.1000 \end{pmatrix}, \quad \mathbf{C}_v = \begin{pmatrix} .1050 & .1965 \\ .1965 & .3946 \end{pmatrix}, \quad \mathbf{C}_a = \begin{pmatrix} .03936 & .06563 \\ .06563 & .1583 \end{pmatrix}.$$

Solving the respective eigenvalue problems we obtain

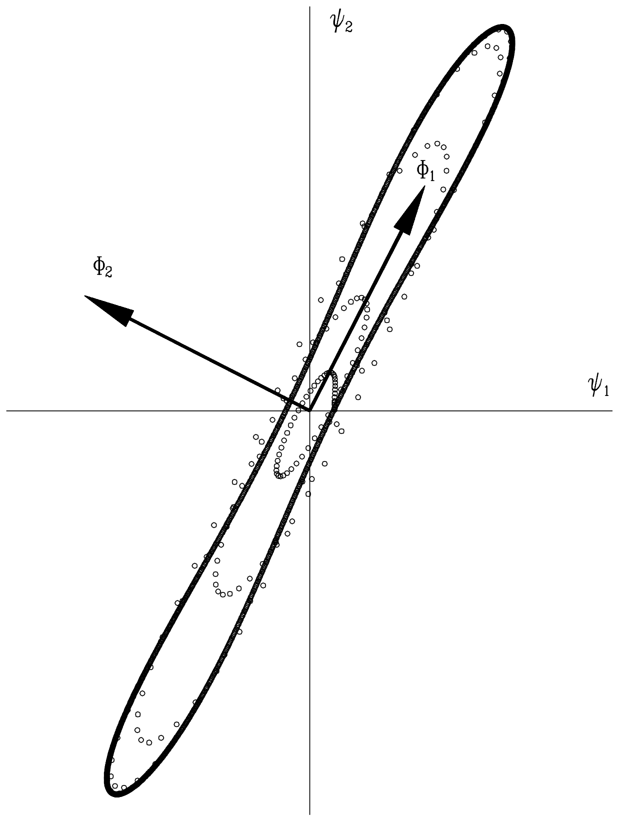
$$\begin{array}{llllllllll} \lambda_1^p & = & 1.386 & \boldsymbol{\phi}^1 & = & \begin{pmatrix} .4559 \\ .8900 \end{pmatrix} & \lambda_2^p & = & .8974 \cdot 10^{-2} & \boldsymbol{\phi}^2 & = & \begin{pmatrix} -.8900 \\ .4559 \end{pmatrix} \\ \lambda_1^v & = & .4939 & \boldsymbol{\phi}^1 & = & \begin{pmatrix} .4510 \\ .8925 \end{pmatrix} & \lambda_2^v & = & .5693 \cdot 10^{-2} & \boldsymbol{\phi}^2 & = & \begin{pmatrix} -.8925 \\ .4510 \end{pmatrix} \\ \lambda_1^a & = & .1874 & \boldsymbol{\phi}^1 & = & \begin{pmatrix} .4053 \\ .9142 \end{pmatrix} & \lambda_2^a & = & .1026 \cdot 10^{-1} & \boldsymbol{\phi}^2 & = & \begin{pmatrix} -.9142 \\ .4053 \end{pmatrix} \end{array} .$$

All three cases show that the flutter motion is strongly coupled and that the first mode is more significant than the second one.

Normalizing the eigenvalues as probabilities:

$$\begin{array}{llllllll} (p) & \lambda_1 & = & .9778 & \lambda_2 & = & .0222 \\ (v) & \lambda_1 & = & .9886 & \lambda_2 & = & .0113 \\ (a) & \lambda_1 & = & .9479 & \lambda_2 & = & .0521 \ . \end{array}$$

The KL-eivenvectors (the optimal eigenvectors) point into the direction of the axes of the energy ellipsoid of the simulation data.



The scattered points represent the transient motion ending in the limit cycle which is given by the full line.

Comparison of various Galerkin approximations

As example we use large amplitude oscillations of the straight downhanging fluid conveying tube.

For the calculation of the critical parameter value ϱ_c cartesian coordinates are introduced and the linearized eigenvalue problem given by

$$\ddot{x}_2 + \alpha_e \dot{x}_2 + x_2^{IV} + \alpha_1 \dot{x}_2^{IV} + 2\sqrt{\beta}\varrho \dot{x}_2'' + \varrho^2 x_2'' - \gamma \left[(1-s)x_2'\right]' = 0$$

with the boundary conditions

$$\begin{array}{rcccl} x_2(0) & = & x_2'(0) & = & 0 \\ x_2''(1) & = & x'''(1) & = & 0\ . \end{array}$$

must be solved.

For $\varrho = \varrho_c = 8.027$, a loss of stability of the downhanging tube by a supercritical Hopf bifurcation takes place.

Various sets of ansatz functions

We compare three different sets of ansatz functions for the Galerkin reduction:

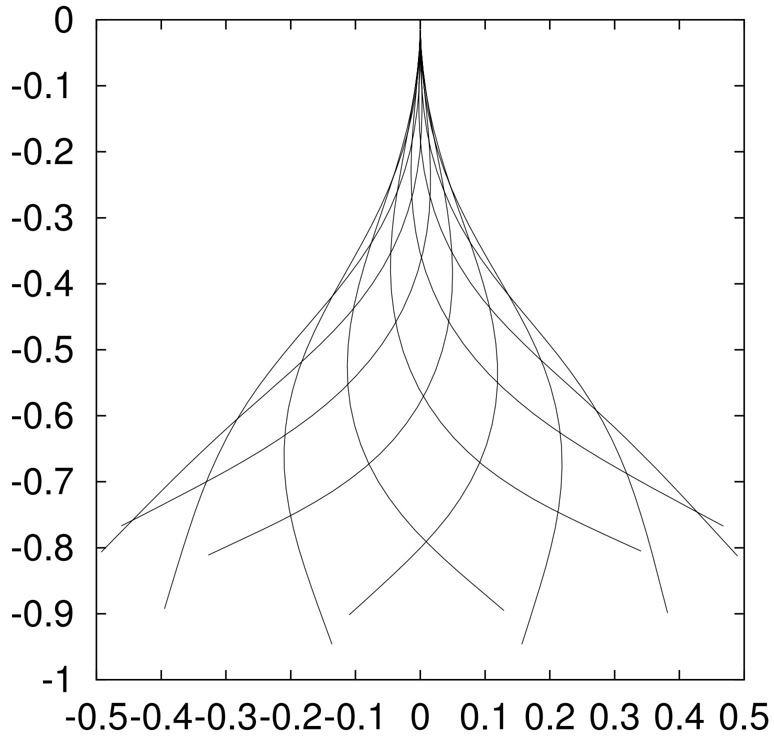
1. Eigenvectors of the linearized problem at $\varrho_s = 10.0 > \varrho_c \approx 8.027$. The eigenvectors are sorted according to the value of the real part of the corresponding eigenvalue.

2. Oscillation modes of a clamped tube (beam). The mode shapes of the unloaded undamped tube (without fluid) are sorted according to their node number.

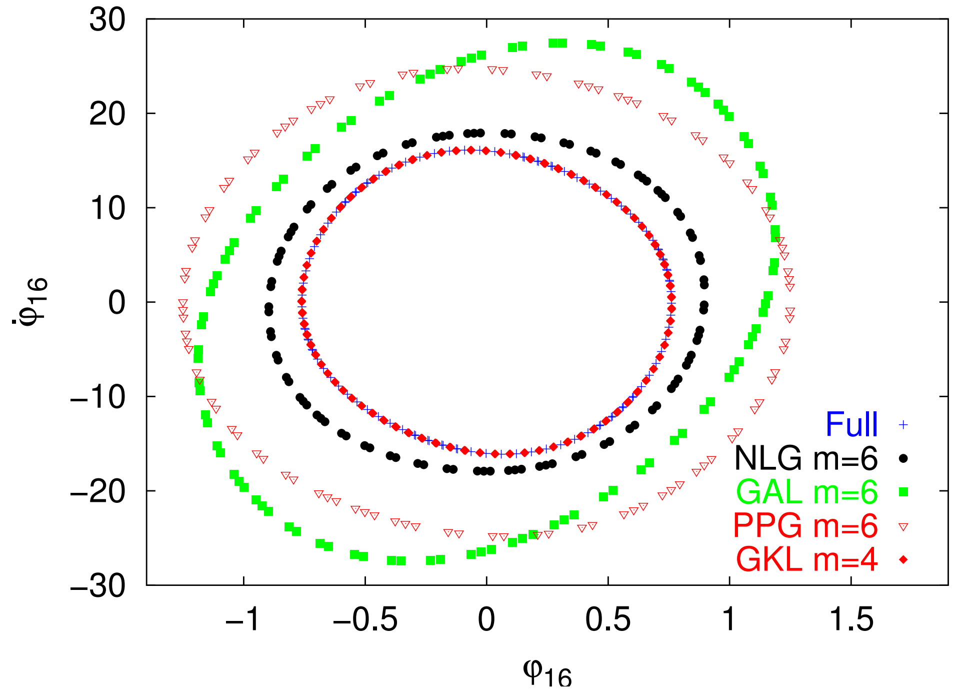
3. Karhunen Loeve basis. The data necessary for the KL-analysis is obtained from the simulation data of the tube motion for $\varrho_s = 10.0$.

Simulation of the tube

Simulation of the fluid conveying tube for $\varrho_s = 10$. A Finite Difference discretisation with $N = 32$ elements is used, resulting in a system with dimension $n = 64$



Projection of the simulation results on the $(\varphi_{16}, \dot{\varphi}_{16})$-plane for: full system $n = 64$ ("Full"), standard Galerkin ("GAL"), nonlinear Galerkin ("NLG"), Galerkin with KL modes ("GKL") and post-processed Galerkin ("PPG"). The results for Full and GKL are not distinguishable.



To see which modes (eigenvectors) make relevant contributions to the motion of the tube we calculate the average contribution of the modes by averaging their amplitudes along the periodic motion. If $\boldsymbol{a}(t)$ is the vector of the coefficients we calculate

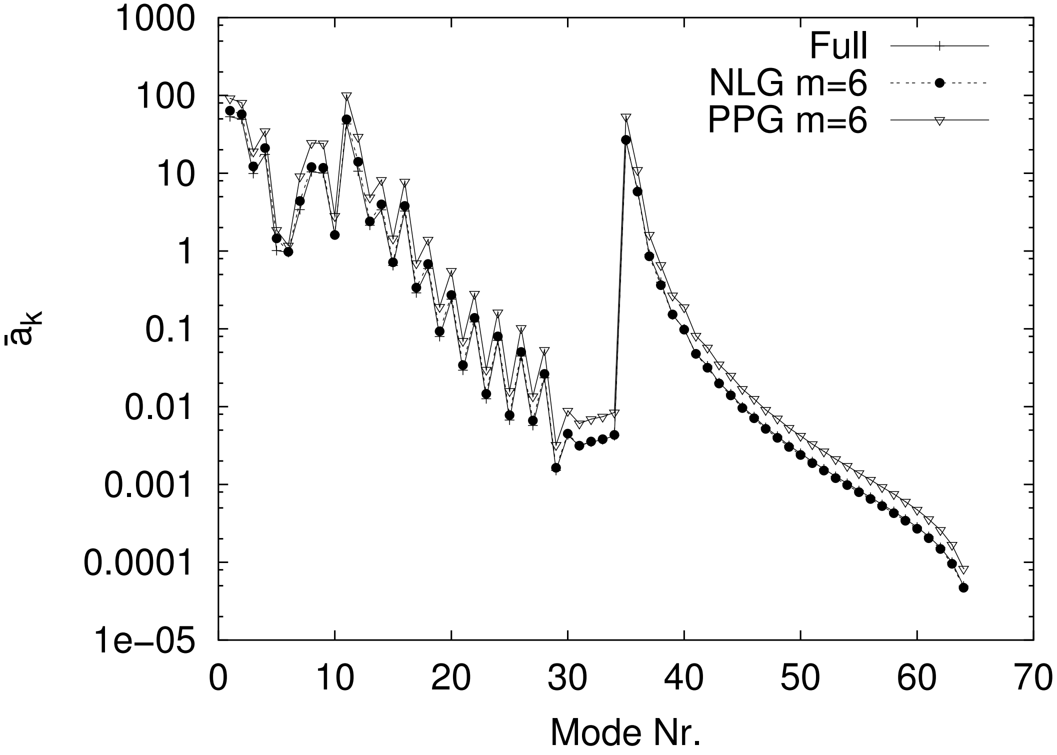
$$\frac{1}{N}\sum_{i=1}^{N} a_k^2(t_i)$$

to obtain the contribution of the k-th mode (eigenvector).

The results are shown for the three different choices of modes in the following sheets

Eigenfunctions

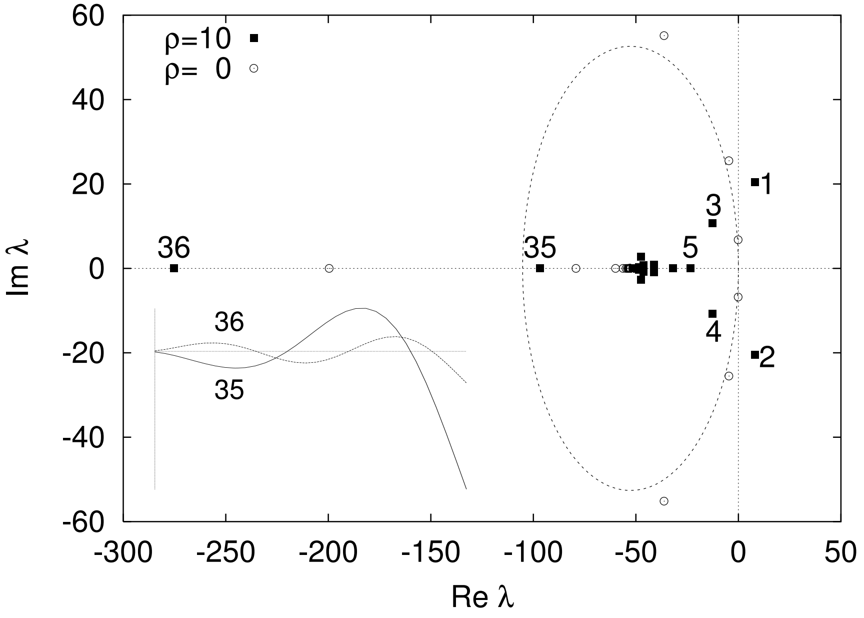
Averaged contribution of each of the first 64 eigenvectors (modes) to the limit cycle oscillation



Distribution of eigenvalues in complex plane

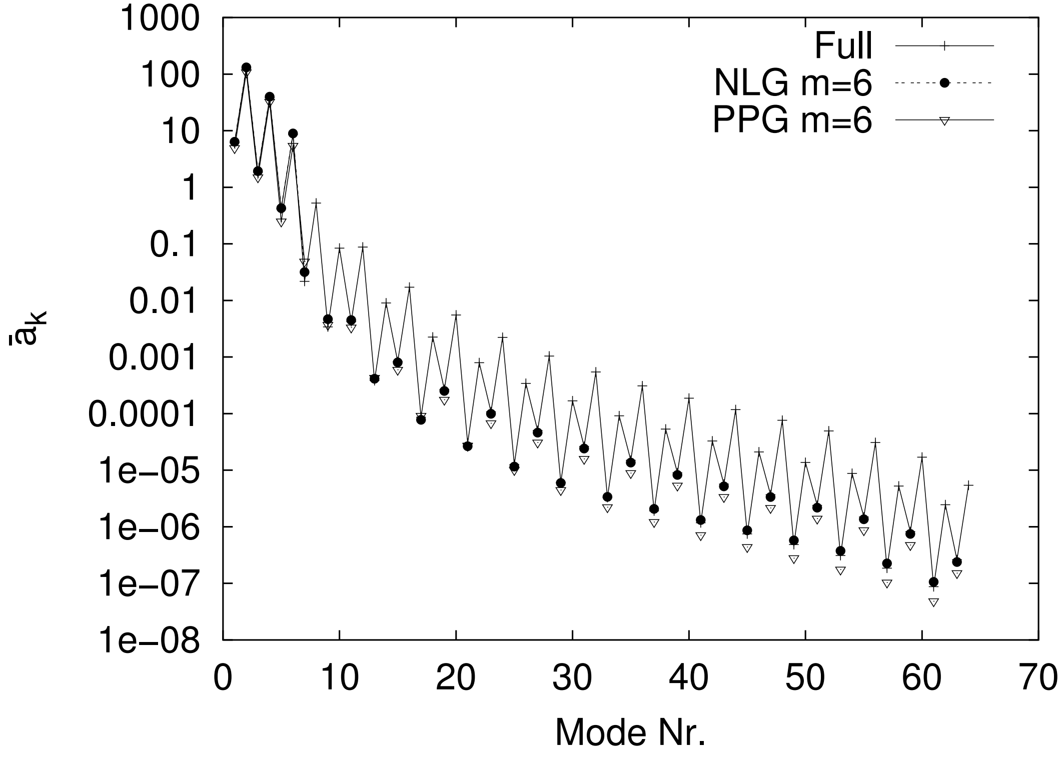
Eigenvalues in the complex plane for $\varrho = 0$ and $\varrho = \varrho_s$ with the modes shapes of eigenvalues 35 and 36.

At $-1/\alpha_1 \approx -50$ (α_1 is the material damping coefficient), an accumulation of the eigenvalues occurs.



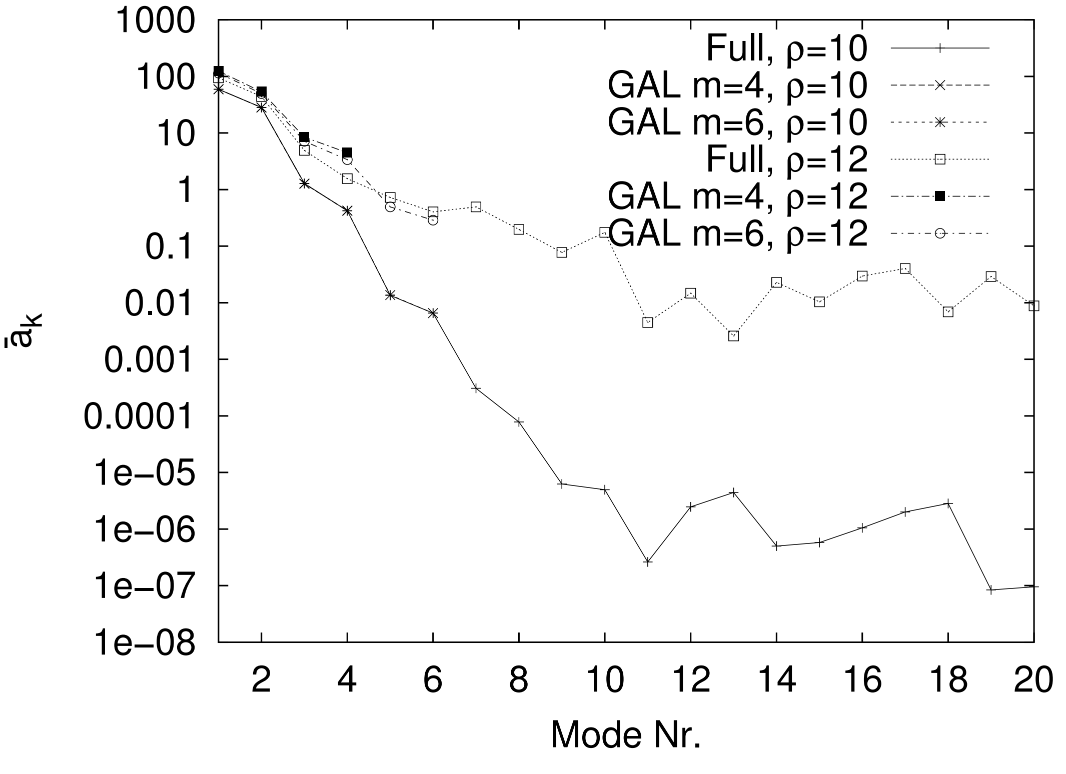
Beam modes

Averaged contribution of the first 64 beam modes to the limit cycle oscillation



KL

Averaged contribution of the KL-modes to the limit cycle oscillation. The results for $\varrho = 12$ are calculated from the simulation data obtained with $\varrho = 10$



Shapes of the first four KL-modes

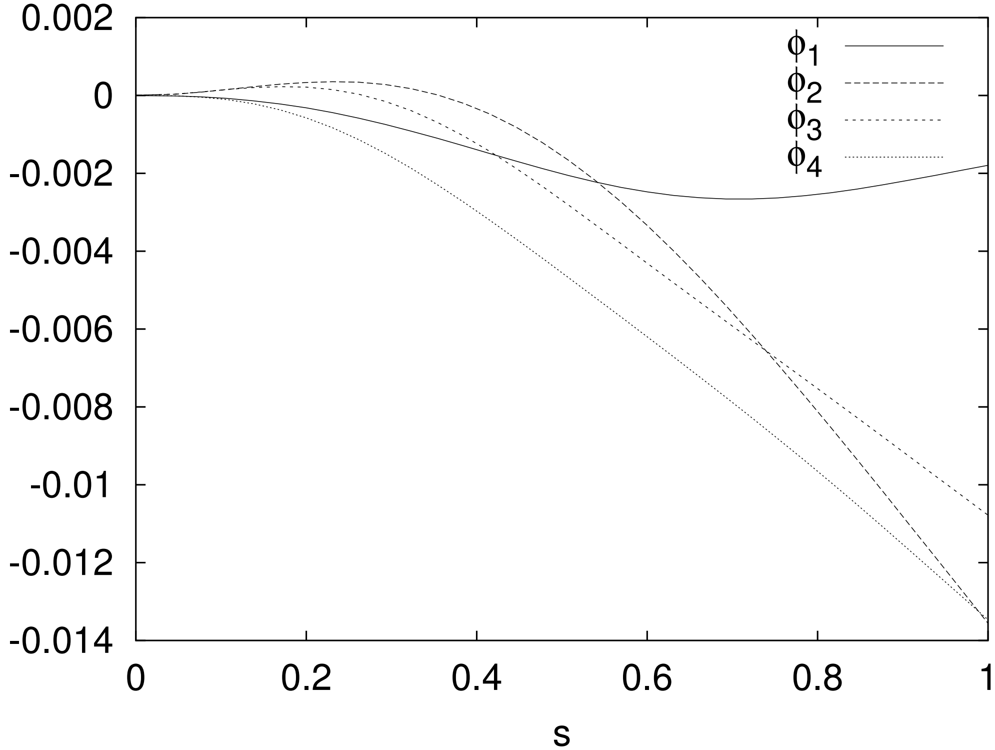


Tabelle 2:

$n = 4$	8	16	32	64
0.7661	0.8049	0.8128	0.8138	0.8147
0.2073	0.1936	0.1867	0.1857	0.1848
0.0265	0.0012	4.64E-4	3.97E-4	3.80E-4
2.2E-5	1.86E-4	5.93E-5	4.49E-5	4.15E-5
	1.66E-7	8.76E-8	4.98E-8	4.30E-8
	1.84E-9	2.26E-8	1.17E-8	9.96E-9
	...	...	...	...

Normalized Eigenvalues of the KL-reduction representing the energy contribution of the respective mode

Condensation (R.J.Guyan, AIAA Journal 1964)

We arrange the structural equation

$$F = Kx$$

so that after partitioning in the form

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the forces F_2 are to be zero. From the second equation $B'x_1 + Cx_2 = 0$ follows

$$x_2 = -C^{-1}B'x_1$$

Hence the variables x_2 at which no forces are applied can be eliminated witout any approximation to result in the reduced system

$$F_1 = (A - BC^{-1}B')x_1$$

with the reduced stiffness matrix

$$K_1 = A - BC^{-1}B'.$$

Condensation is used to reduce the dimension in FE calculations.

Conclusions

1. Invariant manifolds play an important role both for the behaviour of dissipative and conservative systems by slaving fast modes to slow modes.

2. The concept of inertial manifolds is important from a theoretical point of view to show that for certain PDEs a description of the global asymptotic behavior including all attractors can be given by an ODE without any error.

3. In practical calculations Approximate Inertial Manifolds (Nonlinear Galerkin Methods) of various type are used.

4. If a flat Galerkin reduction is used Karhunen-Loeve modes give the best approximation.

5. With arbitrary, admissible, ansatz functions the nonlinear Galerkin via AIM is much

better than the flat Galerkin and allows, what seems to be most important in this respect, a strong dimension reduction.

6. An explanation that the results of the KL approximation for the tube oscillations are so good could be that the limit cycle from which the data was taken will be close to a planar structure. Therefore only two modes already make an essential contribution.

7. For local problems Center Manifold theory works very effective.